

A unified normal mode approach to dynamic tides and its application to rotating Sun-like stars

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ABSTRACT

We determine the response of a uniformly rotating star to tidal perturbations due to a companion. General periodic orbits and parabolic flybys are considered. We evaluate energy and angular momentum exchange rates as a sum of contributions from normal modes allowing for dissipative processes. We consider the case when the response is dominated by the contribution of an identifiable regular spectrum of low frequency modes, such as rotationally modified gravity modes. We evaluate this response in the limit of very weak dissipation, where individual resonances can be significant and also when dissipative effects are strong enough to prevent wave reflection from the neighbourhood of either the stellar surface or stellar centre, making radiation conditions more appropriate. The former situation may apply to Sun-like stars with radiative cores and convective envelopes and the latter to more massive stars with convective cores and radiative envelopes. We provide general expressions for transfer of energy and angular momentum that can be applied to an orbit with any eccentricity.

Detailed calculations require knowledge of the mode spectrum and evaluation of the mode overlap integrals that measure the strength of the tidal interaction. These are evaluated for Sun-like stars in the slow rotation regime where centrifugal distortion is neglected in the equilibrium and the traditional approximation is made for the normal modes. We use both a WKBJ procedure and direct numerical evaluation which are found to be in good agreement for regimes of interest. The former is used to provide expressions for the mode spectrum and overlap integrals as a function of mode frequency and stellar rotation rate. These can be used to find the tidal energy and angular momentum exchange rates and hence the orbital evolution.

Finally we use our formalism to determine the evolution time scales for an object, in an orbit of small eccentricity, around a Sun-like star in which the tidal response is assumed to occur. Systems with either no rotation or synchronous rotation are considered. Only rotationally modified gravity modes are taken into account under the assumption that wave dissipation occurs close to the stellar centre. It is noted that inertial waves excited in the convective envelope may produce a comparable amount of tidal dissipation in the latter case for sufficiently large orbital periods.

Key words: hydrodynamics - celestial mechanics - planetary systems: formation, planet -star interactions, stars: binaries: close, rotation, oscillations, solar-type

1 INTRODUCTION

The tidal interaction between two bodies is an important problem in astrophysics. It plays a role in close binary stars leading to synchronisation and orbital circularisation (eg. Zahn 1977, Hut 1981), as well as stars undergoing a close encounter that may lead to a tidal capture (eg. Press & Teukolsky 1977). In recent years the discovery of many extrasolar planets orbiting in close proximity to their cen-

tral stars has highlighted another situation where tidal interactions may play an important role in the formation and evolution of the systems (eg. Barker & Ogilvie 2009). In particular hot Jupiters may be formed by a tidal capture into a highly eccentric orbit followed by orbital circularisation (Weidenschilling & Marzari 1996, Rasio & Ford 1996, Nagasawa, Ida & Bessho 2008). During this process tidal dissipation both in the planet and the star can be significant (Ivanov & Papaloizou 2004, 2008, 2011).

The general problem of determining the tidal evolution of a system involves finding the response of the rotating stars and/or planets due to the gravitational perturbation of a companion. This results in the excitation of normal modes and the exchange of en-

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ergy and angular momentum with the orbit leading to its evolution. This may be technically challenging on account of the normal mode spectrum being dense and/or continuous (Papaloizou & Pringle 1982). Critical latitude phenomena and wave attractors may also be involved in some cases (Ogilvie & Lin 2004). Multidimensional tidal response calculations need to be undertaken for bodies in a variety of evolutionary states and rates of rotation. Accordingly it is useful to formulate methods for calculating the tidal interaction that involve the evaluation of manageable analytic expressions involving functions of at most one variable that can be readily tabulated.

In this paper we develop general procedures for expressing tidal energy and angular momentum exchange rates in simplified terms that are applicable to the tidal response that arises from an identifiable regular spectrum of low frequency modes, comprised of for example rotationally modified gravity modes, for rotating stars with realistic structure. These are likely to give the dominant tidal response in bodies with stratification, where the tidal forcing frequencies significantly exceed the inverse of the convective time scale associated with any convection zone, making the associated turbulent viscosity inefficient. The waves associated with the rotationally modified gravity modes may either produce a long lasting resonant response through the so-called 'mode locking' mechanism (eg. Witte & Savonije 1999, 2002), or be dissipated, either near the centre in Sun-like stars with a radiative core and convective envelope (eg. Barker 2011), or near the surface in massive stars with a convective core and radiative envelope (eg. Zahn 1977).

When decomposition of free stellar pulsations over normal modes is possible, and the damping mechanism of normal modes is specified, we give general expressions from which the energy and angular momentum exchange may be calculated, in principle, for both bound orbits of arbitrary eccentricity and parabolic encounters that could lead to tidal captures with an arbitrary inclination of stellar rotation axis with respect to orbital plane. These are expressed in terms of contributions from individual normal modes. In general, they require evaluation of coefficients characterising the representation of the tidal potential as either a Fourier series or a Fourier transform, the spectrum of normal modes, normal mode damping rates and so called overlap integrals that measure the strength of interaction. It is important to note that in certain regimes of tidal interaction, such as energy and angular momentum exchange through parabolic encounters, or orbital evolution in the regime of so-called moderately strong viscosity (eg. Zahn 1977, Goodman & Dickson 1998), the resulting expressions do not depend on the details of the mode damping, which is poorly known in many situations, see also the discussion of these regimes below.

The overlap integrals determine the coupling of mode eigenfunctions with the tidal potential, they thus play a central role in our approach to tidal problem. Note that although we use the definition of overlap integral first introduced by Press & Teukolsky (1977) for non-rotating stars and later generalised by Ivanov & Papaloizou (2005, 2007) for rigidly rotating stars, they are directly related to the so-called tidal resonance coefficients first introduced by Cowling (1941) and studied later in a number of papers (eg. Zahn 1970, Rocca 1987, Smeyers, Willems & Van Hoolst 1998). In order to calculate the overlap integrals one should be able to evaluate integrals with highly oscillatory integrands over the bulk of the star. This is rather difficult to achieve both numerically and analytically. In such a situation, numerical and analytical methods should supplement each other. In this paper we give an analytic method for their evaluation, together with a calculation of the relevant eigen-spectrum and normal modes, valid for Sun-like stars having an in-

ner radiative region and a convective envelope, that is based on a WKBJ scheme. In an accompanying paper we also provide values of overlap integrals for a range of eigenfrequencies in a number of models of more massive stars of different masses and ages.

It is quite important to stress that the calculation of the relevant normal mode spectra for rotating stars is complex, because unlike the situation for a spherical body, separation of variables is not in general possible. However, this is rectified if the well known traditional approximation is used as is done below. In the traditional approximation only the radial component of the angular velocity is retained (Unno et al. 1989). This is expected to be a good approximation for rotationally modified gravity modes for which motions are mainly horizontal. It has accordingly been adopted by many authors. For example, it was used in an asymptotic treatment of the normal mode problem by Berthomieu et al. (1978) and in direct tidal response calculations (Papaloizou & Savonije 1997, Witte & Savonije 1999, 2002 and Ogilvie & Lin 2004). However, it is worth mentioning that the existence of regular spectra of rotationally modified gravity modes is not dependent on the use of the traditional approximation. Such spectra are expected to be associated with the radiative cores of Sun-like stars and the radiative envelopes of more massive stars. A spectrum of rotationally modified gravity modes in the case of a massive slowly rotating star was identified by Savonije et al. (1995). In this case the traditional approximation was found to be a good one as long as their eigenfrequencies placed them outside the inertial regime (the absolute value of the eigenfrequency exceeded twice the stellar angular velocity). Similarly Dintrans & Rieutord (2000) found a rotationally modified gravity mode spectrum outside the inertial regime for a $1.5M_{\oplus}$ star with a convective envelope without making use of the traditional approximation. Note too that the order of the $l = 2$ g mode with eigenfrequency corresponding to the critical break up rotation frequency for a Sun-like star is typically ~ 10 . Accordingly a low frequency asymptotic analysis should be useful for stars in a state of modest rotation, at less than about one fifth of the critical rate, and excited modes with comparable eigenfrequencies.

Finally, we demonstrate how our approach can be applied to the calculation of the orbital circularisation and decay time scales for objects orbiting Sun-like stars under the assumption of wave dissipation near the centre and small orbital eccentricity.

The plan of this study is similar to that of our previous study of inertial waves in rotating bodies undertaken in Ivanov & Papaloizou (2011) and Papaloizou & Ivanov (2011). In this paper we mainly concentrate on an analytic approach to the problem while in an accompanying paper we consider the application of numerical methods, extending our calculations to evaluate the tidal response of stellar models appropriate to stars more massive than the sun.

In section 2 we give the basic equations governing the tidal response of a rotating star to tidal forcing. We then use them to calculate the linear response to harmonic forcing for a particular azimuthal mode number in section 3. To do this we apply an extension of the method of Ivanov & Papaloizou (2007), hereafter IP7, which provides a solution in terms of a decomposition in terms of normal modes, that allows us to incorporate dissipative forces. Specifically we consider viscous forces for which the viscosity coefficient may vary with location but the approach may be generalised to deal with more general forms of dissipation.

We focus on the case where there is a regular spectrum of normal modes that can be dense, such as a low frequency gravity mode spectrum and consider its response. In particular we calculate the rate of transfer of energy and angular momentum between the star and the orbit, expressing the results in terms of overlap integrals

that measure the contribution associated with each normal mode in terms of the projection of a related eigenfunction onto the tidal potential. We then customise the expressions for the tidally induced energy and angular momentum exchange rates to apply to the case where the response is determined by a dense regular set of normal modes, such that the mode decay rates are small compared to their natural frequencies, but not small compared to the separation between adjacent mode frequencies.

As we demonstrate in an appendix for slow rotation under the traditional approximation, this case is expected to correspond to the situation where waves associated with the modes propagate to localised boundary regions, which may be either close to the stellar centre or surface, where they are dissipated, and we call it the moderately viscous case. This is in contrast to the situation where dissipation is very weak, and the interaction is dominated by strong normal mode resonances that are effective only in very narrow frequency intervals (eg. Savonije et al. 1995, Witte & Savonije 1999, 2002). The expressions we obtain can be used to determine the orbital evolution of a binary system, in a general periodic orbit which may be near circular, or undergoing encounters in a near parabolic orbit, induced by tides. Knowledge of the adiabatic normal mode spectrum and the overlap integrals corresponding to each mode, in the low frequency regime, is required for their evaluation in the moderately viscous case, the results then being independent of the details of the dissipation process.

In section 4 we go on to discuss the linear adiabatic normal mode problem and the low frequency spectrum provided by rotationally modified gravity modes. We consider the low frequency asymptotic regime under the traditional approximation in which only the radial component of the angular velocity is retained. When this approximation is made, separation of variables is possible. Its efficacy in the stellar case was first implicitly realised by Berthomieu et al. (1978), through the application of an asymptotic procedure, that effectively led to it. We develop expressions for the overlap integrals in the low frequency limit that can be evaluated as one dimensional integrals involving the mode angular and radial eigenfunctions.

In section 5 we perform direct numerical calculations of mode eigenfrequencies, eigenfunctions and their associated overlap integrals. Rotationally modified gravity modes under the traditional approximation are found for models of solar-type stars. Models representing both the present day sun and a young sun with age 1.66×10^8 yr. are considered. We also give power fits to the dependence of the square of the Brunt - Väisälä frequency inside the radiative core, on the distance to the boundary between the interior radiative zone and the convective envelope (see also Barker 2011). This dependence was found to have a significant effect on properties of the normal modes.

In section 6 we calculate eigenfrequencies, eigenfunctions and overlap integrals for high order adiabatic rotationally modified gravity modes for models of solar-type stars under the traditional approximation following a WKBJ approach that incorporates some corrections to the formalism of Berthomieu et al. (1978). We also allow for an arbitrary power law dependence of the Brunt - Väisälä frequency in the radiative interior on the distance to the convective envelope boundary. We provide expressions for both eigenfunctions and eigenvalues which are found to provide a good approximation to the corresponding numerically determined quantities for the stellar models considered.

We use these solutions to evaluate the overlap integrals, providing analytic expressions for them, which together with some quantities represented graphically, readily enable their evaluation

as a function of mode frequency and rotation rate of the star. These are in very good agreement with the numerically determined quantities for mode frequencies less than the critical rotation frequency.

Knowledge of the overlap integrals and mode spectrum enables calculation of tidal energy and angular momentum exchange rates with a general orbit. In section 7 we apply our formalism to the calculation of the tidal evolution timescales for a binary with a small eccentricity and with aligned orbital and spin angular momenta. One of the components is assumed to be a Sun-like star in which modified gravity modes are excited and propagate to the centre where they are assumed to be dissipated corresponding to the moderately large viscosity regime. The other component is treated as a point mass and a large range of mass ratios is considered. We give estimates for the circularisation time scale when the star is non rotating and when it rotates synchronously with the orbit. In the former case we give the time scale for the semi-major axis to decrease.

Finally in section 8 we summarize and discuss our results.

2 BASIC EQUATIONS

2.1 Coordinate system and notation

The basic definitions and notation adopted in this paper closely follow IP7. We use either cylindrical coordinate system (ϖ, ϕ, z) or spherical (r, ϕ, θ) coordinate systems with origin at the centre of mass of the star. When viewed in an inertial frame with this origin, the unperturbed star is assumed to rotate uniformly about the z axis with angular velocity Ω . Accordingly we adopt the rotating frame in which the unperturbed star appears at rest as our reference frame.

In this paper the fundamental quantity describing perturbations of a rotating star will be the Lagrangian displacement vector ξ . By definition, this is a real quantity. We represent the displacement vector as well as any other perturbation quantity in terms of a Fourier series in the azimuthal angle ϕ , and either in terms of a Fourier series in the time, t , or a Fourier integral with respect to t , depending respectively on whether we consider tidal interactions in a binary having a periodic orbit, or the orbit is assumed to be parabolic.

When a discrete Fourier series in time is appropriate, for a real quantity, Q , we write

$$Q = \sum_{m,k} (Q_{m,k} \exp(-i\omega_{m,k}t + im\phi) + cc), \quad (1)$$

where cc denotes the complex conjugate, $\omega_{m,k} = k\Omega_{orb} - m\Omega$, Ω_{orb} is the orbital frequency, Ω is the angular velocity of the star and m and k are integers, with only positive values of m included in the summation as in IP7. Note when the orbital plane is perpendicular to the rotational axis of the star, and only the quadrupole component of the tidal potential, that is dominant at large separations is included, terms with $m = 1$ are absent in the summation. It then suffices to consider $m = 0, 2$. In addition, the reality of Q implies that we should identify $Q_{-m,-k} = Q_{m,k}^*$. When forming the sum (1), we ensure that terms are not repeated. For example, when $m = 0$, terms for which $\omega_{m,k}$ changes sign are identical, they are accordingly combined. After this is done, forcing for only one sign of $\omega_{m,k}$ has to be considered.

For the case involving a Fourier transform, we write

$$Q = \sum_m \left(\exp(im\phi) \int_{-\infty}^{+\infty} d\sigma \tilde{Q}_m \exp(-i\sigma t) + cc \right), \quad (2)$$

where the Fourier transform is indicated by a tilde. This requirement that Q is real leads to the identification $\tilde{Q}_{-m}(-\sigma) = \tilde{Q}_m^*(\sigma)$. The inner products of two complex scalars Y_1, Y_2 , and two complex vectors $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ are respectively defined through

$$(Y_1|Y_2) = \int \varpi d\varpi dz \rho (Y_1^* Y_2) \quad \text{and} \quad (3)$$

$$(\boldsymbol{\eta}_1|\boldsymbol{\eta}_2) = \int \varpi d\varpi dz \rho (\boldsymbol{\eta}_1^* \cdot \boldsymbol{\eta}_2), \quad (4)$$

where $(\boldsymbol{\eta}_1^* \cdot \boldsymbol{\eta}_2)$ is the scalar product, ρ is the density, and $*$ stands for the complex conjugate.

2.2 Equations of motion

In the rotating frame the linearised Navier-Stokes equations take the form (see IP7)

$$\ddot{\boldsymbol{\xi}} + 2\boldsymbol{\Omega} \times \dot{\boldsymbol{\xi}} + \mathcal{C}\boldsymbol{\xi} = -\nabla\Psi - \mathbf{f}_\nu, \quad (5)$$

where the dot indicates partial differentiation with respect to time and \mathcal{C} is an integro-differential operator describing the action of gravity and pressure forces. For conservative boundary conditions, it is self-adjoint when the normal product (4) is adopted, and it can be considered to be a positive operator under certain conditions, which are assumed to be fulfilled in this paper. Its general explicit form is not important for our purposes and can be found in e.g. Lynden-Bell & Ostriker 1967.

Retaining only the assumed dominant quadrupole component, the tidal potential, Ψ can be represented in the form

$$\Psi = r^2 \sum_{m,k} (A_{m,k} e^{-i\omega_{m,k} t} Y_2^m + \text{cc}) \quad (6)$$

where Y_2^m are the spherical functions. The coefficients $A_{m,k}$ are given in Appendix A for the case of coplanar orbit with small eccentricity. For orbits having an arbitrary value for the eccentricity, these coefficients can be expressed in terms of so-called Hansen coefficients discussed eg. in Witte & Savonije (1999). However, note that the representation of the tidal potential in terms of a Fourier series (6) is not convenient for a highly eccentric orbit. In that case it is more appropriate to adopt the tidal potential for a parabolic orbit, and its Fourier transform. The corresponding expressions can be found in e.g. Ivanov & Papaloizou (2011). These authors as well as Lai & Wu 2006 also give expressions, which can be used to represent the tidal potential in a coordinate system that is rotated with respect to the one for which the z axis is normal to the orbital plane. These expressions can be used when the rotation axis of the star is not perpendicular to the orbital plane.

The viscous force $\mathbf{f}_\nu = \boldsymbol{\sigma}\dot{\boldsymbol{\xi}}$, where the operator $\boldsymbol{\sigma}$ is defined through its action on a vector $\boldsymbol{\eta}$ as

$$(\boldsymbol{\sigma}\boldsymbol{\eta})_\alpha = -\frac{1}{\rho}(\rho\nu\sigma_{\alpha\beta})_{,\beta}, \quad \sigma_{\alpha\beta} = \eta_{\alpha,\beta} + \eta_{\beta,\alpha} - \frac{2}{3}\eta_{\gamma,\gamma}\delta_{\alpha\beta} \quad (7)$$

where ν is kinematic viscosity, a comma stands for differentiation over Cartesian coordinates and summation over repeating indices is implied. Note that we reserve the indices α, β and γ to indicate components of vectors and tensors in order to distinguish them from indices used in the summation of series. When these indices are repeated, summation is implied from now on (the summation convention).

We remark that we have considered a standard form of viscous dissipation with the possibility that the viscosity coefficient depends on position. This leads to some simplifying features in the

discussion of dissipation below on account of the viscous stress tensor being symmetric. However, a corresponding analysis can also be undertaken for more complicated forms of dissipation, including radiative damping, by introducing relevant adjoint operators. However, as the final results we use do not depend on the details of the dissipation beyond involving decay rates associated with individual normal modes we shall work only with the standard formulation for viscous dissipation described above.

3 THE RESPONSE TO TIDAL FORCING DUE TO COMPANIONS IN PERIODIC AND PARABOLIC ORBITS

In this section we calculate the tidal response to tidal forcing associated with a general periodic orbit. Later we extend the discussion to the limiting case where the orbit becomes parabolic (see IP7). We begin by writing the forcing potential (6) as a Fourier series of the form (1). The Fourier coefficients, $\psi_{m,k}$, are such that

$$(\nabla\Psi)_{m,k} = \frac{A_{m,k}}{2\pi} \int_0^{2\pi} d\phi e^{-im\phi} \nabla(r^2 Y_2^m), \quad (8)$$

where Y_2^m indicates the usual spherical harmonic with indices m and l .

As was shown by Papaloizou & Ivanov 2005 and IP7, after taking the Fourier transform of equation (5), one can obtain equations for determining the Fourier coefficients of the Lagrangian displacement $\boldsymbol{\xi}_{m,k}$. These can be organised into a convenient form as described below. Once determined for periodic orbits, the $\boldsymbol{\xi}_{m,k}$ can then be used to calculate the orbital energy and angular momentum exchange rates.

3.1 Determination of the Fourier components of the Lagrangian displacement

We begin by introducing a six-dimensional vector $\vec{Z}_{m,k} \equiv (\mathbf{Z}_1, \mathbf{Z}_2)^T$ where

$$\mathbf{Z}_1 = \omega_{m,k} \boldsymbol{\xi}_{m,k}, \quad \text{and} \quad \mathbf{Z}_2 = \mathcal{C}^{1/2} \boldsymbol{\xi}_{m,k}. \quad (9)$$

$$\text{Equation (5) then follows from the pair of equations derived from } \omega_{m,k} \vec{Z}_{m,k} = \mathcal{H} \vec{Z}_{m,k} - i\omega_{m,k} \mathcal{D} \vec{Z}_{m,k} + \vec{S}, \quad (10)$$

where the operator \mathcal{H} has a matrix structure

$$\mathcal{H} = \begin{pmatrix} \mathcal{B} & \mathcal{C}^{1/2} \\ \mathcal{C}^{1/2} & 0 \end{pmatrix}, \quad (11)$$

with the operator $\mathcal{C}^{1/2}$ being defined through the condition: $\mathcal{C} = \mathcal{C}^{1/2}\mathcal{C}^{1/2}$, with any ambiguity removed by requirement that it is a positive operator (see e.g. Wouk 1966). In addition the operator

$$\mathcal{B}\boldsymbol{\xi} = -2i\boldsymbol{\Omega} \times \boldsymbol{\xi}. \quad (12)$$

The operator \mathcal{D} is defined through

$$\mathcal{D}\vec{Z}_{m,k} = (\mathbf{f}_{\nu 1,k}, \mathbf{f}_{\nu 2,k})^T, \quad (13)$$

where

$$\mathbf{f}_{\nu 1,k} = \frac{1}{\omega_{m,k}} \boldsymbol{\sigma} \mathbf{Z}_1 \quad \text{and} \quad \mathbf{f}_{\nu 2,k} = 0. \quad (14)$$

As we have performed a Fourier decomposition in terms of m , the action of the operator \mathcal{O} being any one of \mathcal{B}, \mathcal{C} or $\boldsymbol{\sigma}$ on the Fourier component $\boldsymbol{\xi}_{m,k}$ is obtained through

$$\mathcal{O}\xi_{m,k} = \exp(-im\phi)\mathcal{O}(\xi_{m,k} \exp(im\phi)). \quad (15)$$

It readily follows that \mathcal{B} is self-adjoint as are \mathcal{C} and $\mathcal{C}^{1/2}$ (IP7). From the definition (7) it also follows that σ is self-adjoint. The six dimensional source vector $\vec{S} \equiv (\mathbf{S}_1, \mathbf{S}_2)^T$ where

$$\mathbf{S}_1 = (\nabla\Psi)_{m,k} \text{ and } \mathbf{S}_2 = 0. \quad (16)$$

Equation (10) can be used to define an eigenvalue problem in the form

$$\lambda_j \vec{Z}^j = (\mathcal{H} - i\omega_{m,k} \mathcal{D}) \vec{Z}^j. \quad (17)$$

Here λ_j is the generally complex eigenvalue and the corresponding eigenfunction is $\vec{Z}^j = (\mathbf{Z}_{1j}, \mathbf{Z}_{2j})^T$, the components being related to the corresponding displacement by

$$\mathbf{Z}_{1j} = \lambda_j \xi_j, \text{ and } \mathbf{Z}_{2j} = \mathcal{C}^{1/2} \xi_j. \quad (18)$$

Adopting the inner product defined through

$$\langle \vec{X} | \vec{Z} \rangle = \int \varpi d\varpi dz \rho(\mathbf{X}_1^*, \mathbf{X}_2^*) (\mathbf{Z}_1, \mathbf{Z}_2)^T, \quad (19)$$

it follows from equations (11) and (13) that the operators \mathcal{H} and \mathcal{D} are self-adjoint. Hence it follows that the operator $\mathcal{H} + i\omega_{m,k} \mathcal{D}$ is the adjoint operator associated with the eigenvalue problem (17). The eigenvalues will accordingly be eigenvalues λ_j^* with corresponding eigenfunctions \vec{X}^j .

In addition the eigenfunctions \vec{Z}^j and \vec{X}^j are biorthogonal with respect to the adopted inner product so that

$$\langle \vec{X}^j | \vec{Z}^l \rangle = \lambda_j^* \lambda_l (\eta_j | \xi_l) + (\eta_j | \mathcal{C} \xi_l) = \delta_{jl} N_j, \quad (20)$$

where η_j denotes the Lagrangian displacement associated with the eigenfunctions of the adjoint problem defined through (18) with \mathbf{Z} being replaced by \mathbf{X} . The norm is

$$N_j = \langle \vec{X}^j | \vec{Z}^j \rangle = |\lambda_j|^2 (\eta_j | \xi_j) + (\eta_j | \mathcal{C} \xi_j). \quad (21)$$

We remark that although we have not assumed the viscosity is small up to now, when it is neglected ξ_j and η_j coincide.

3.2 Solution for the forced response

We look for solution of (10) as a decomposition over the eigenvectors \vec{Z}^j :

$$\vec{Z}_{m,k} = \sum_j \alpha_j \vec{Z}^j. \quad (22)$$

Substituting (22) in (10), taking the inner product of the result with the eigenvector \vec{X}^l and using (17) and (20) we get

$$\alpha_l (\omega_{m,k} - \lambda_l) N_l = \langle \vec{X}^l | \vec{S} \rangle. \quad (23)$$

To proceed further we remark that an expression for the damping rate of the mode with eigenvalue λ_j can be found by taking the inner product of (17) with \vec{Z}^j . Using the fact that \mathcal{H} is self-adjoint, we find that

$$\mathcal{I}m(\lambda_j) \langle \vec{Z}^j | \vec{Z}^j \rangle = -\omega_{m,k} \langle \vec{Z}^j | \mathcal{D} \vec{Z}^j \rangle, \quad (24)$$

where $\mathcal{I}m$ denotes the imaginary part. Setting $\mathcal{I}m(\lambda_j) = -\omega_{\nu,kj}$ and making use of (18), (13) and (7) we obtain

$$\omega_{\nu,kj} = \frac{|\lambda_j|^2}{2N_j} (\nu \sigma_{\alpha\beta}^j | \sigma_{\alpha\beta}^j), \quad (25)$$

where the superscript j indicates that $\sigma_{\alpha\beta}^j$ is evaluated assuming that η as used in equation (7) is replaced by ξ_j and

$$N_j = |\lambda_j|^2 (\xi_j | \xi_j) + (\xi_j | \mathcal{C} \xi_j). \quad (26)$$

Using the above we write $\lambda_j = \omega_j - i\omega_{\nu,kj}$, where ω_j is the real part of λ_j . We remark that for convenience in the above discussion we have suppressed the index k for λ_j , ω_j and ξ_j . Then from equation (23) we find

$$\alpha_j = \frac{\lambda_j S_{j,k}}{N_j (\omega_{m,k} - \omega_j + i\omega_{\nu,kj})}, \quad (27)$$

where

$$S_{j,k} = (\eta_j | \nabla \Psi_{m,k}), \quad (28)$$

Now we use equation (22) to obtain

$$\xi_{m,k} = \sum_j \frac{\lambda_j^2 S_{j,k}}{\omega_{m,k} N_j (\omega_{m,k} - \omega_j + i\omega_{\nu,kj})} \xi_j. \quad (29)$$

3.3 Energy transfer rate due to tidal forcing

The canonical energy associated with the perturbations is given by a standard expression

$$\begin{aligned} E_c &= \int_V d^3 x \rho \frac{1}{2} (|\dot{\xi}|^2 + \xi \cdot \mathcal{C} \xi) \text{ which leads to} \\ \langle E_c \rangle &= 2\pi \sum_{m,k} \{ \omega_{m,k}^2 (\xi_{m,k} | \xi_{m,k}) + (\xi_{m,k} | \mathcal{C} \xi_{m,k}) \}, \end{aligned} \quad (30)$$

where we integrate over the volume of the star. Recall that ξ is real. The last expression, being a time average, displays explicitly contributions corresponding to different azimuthal numbers m and forcing frequencies $\omega_{m,k}$. The contribution of an equal response to the complex conjugate of the forcing potential in (6) has also been included. The rate of change of E_c with time as a result of the action of dissipative forces follows from (5) and (30) as

$$\dot{E}_c = - \int d^3 x \rho (\dot{\xi} \cdot \mathbf{f}_\nu) = -\frac{1}{2} \int d^3 x \rho \nu [\dot{\sigma}_{\alpha\beta} \dot{\sigma}_{\alpha\beta}], \quad (31)$$

where we use (7) with η replaced by ξ and integrate by parts. We remark that when, as here a sum of steady periodic responses is excited by a forcing potential, the rate of energy loss resulting from dissipative forces is balanced by input from the forcing potential. We now go on to obtain an expression for \dot{E}_c in the form of a sum of contributions corresponding to different m and $\omega_{m,k}$.

To do this we substitute (29) in the appropriate form of (1) and the result into (31). An average of the resulting expression over a time period that is large in comparison to the periods of eigenmodes that contribute significantly to the sum is then taken. We thus obtain

$$\dot{E}_c = -\pi \sum_{m,k,j,j'} \left(\frac{\lambda_j^2 \lambda_{j'}^{*2} S_{j,k} S_{j',k}^*}{N_j N_{j'}^* D_{k,j,j'}} (\nu \sigma_{\alpha\beta}^{j'} | \sigma_{\alpha\beta}^j) + cc \right), \quad (32)$$

where

$$D_{k,j,j'} = (\omega_{m,k} - \omega_j + i\omega_{\nu,kj}) (\omega_{m,k} - \omega_{j'} - i\omega_{\nu,kj'}). \quad (33)$$

When the mode energy does not grow with time on average, and, accordingly, the stationarity condition is implied as in the bulk of the paper¹ the rate of energy dissipation by the action of viscosity must be negative of the work done by tidal forces, $\dot{E}_c = -\dot{E}_T$, where $\dot{E}_T = -2\pi (\dot{\xi} | \nabla \Psi)$, see equation (5) and averaging over

¹ We briefly consider the situation when the stationarity condition may be invalid in the end of Discussion.

a time larger than a characteristic time scales associated with the forcing frequencies is implied. From equations (1), (9) and (22) it is easy to see that

$$\dot{E}_T = 2\pi i \sum_{m,k,j} \omega_{m,k} (\alpha_j ((\nabla \Psi)_{m,k} | \xi_{m,k}) - cc). \quad (34)$$

Note that when viscosity is small the complex conjugate of $(\xi_{m,k} | (\nabla \Psi)_{m,k})$ is approximately given by equation (28).

3.3.1 The case of very small viscosity and near normal mode resonances

The expression (32) can be further simplified by assuming that a strong resonance exists in the vicinity of a particular forcing frequency, $\omega_{m,k}$, on account of very weak dissipation. If we assume as a consequence of this that, for a given forcing frequency, the dominant contribution to the sum comes from a particular mode that is nearly resonant, only the diagonal terms in the sum over j and j' need to be retained as they provide this contribution. But note that the appropriate damping rates due to viscosity should be sufficiently small, as indicated further below. Then we have

$$\begin{aligned} \dot{E}_c &= -2\pi \sum_{m,k,j} \left(\frac{|\lambda_j|^4 |S_{j,k}|^2}{|N_j|^2 D_{k,j,j}} (\nu \sigma_{\alpha\beta}^j | \sigma_{\alpha\beta}^j) \right) \\ &= -4\pi \sum_{m,k,j} \omega_{\nu,kj} |\lambda_j|^2 \mathcal{N}_j \frac{|S_{j,k}|^2}{|N_j|^2 D_{k,j,j}}, \end{aligned} \quad (35)$$

where we have used equation (25) to obtain the last equality.

3.3.2 The case of moderately large viscosity

In order to make the above procedure valid, the effect of viscosity should be small. But note that we shall be interested in the case when the viscous decay rate of a mode, although being small compared to its frequency, may be large compared to the frequency separation between modes. This situation is less clear as many modes could interact and treatment of viscosity by straightforward perturbation theory is not possible. However, taking the inner product of equation (17) with $\tilde{Z}^{j'}$ and subtracting the complex conjugate of the corresponding equation with j and j' interchanged, we obtain

$$(\lambda_j - \lambda_{j'}^*) < \tilde{Z}^{j'} | \tilde{Z}^j > = -2i\omega_{m,k} < \tilde{Z}^{j'} | \mathcal{D} \tilde{Z}^j > \quad (36)$$

which leads to the following expression

$$\frac{\lambda_j \lambda_{j'}^* (\nu \sigma_{\alpha\beta}^j | \sigma_{\alpha\beta}^j)}{(\lambda_j - \lambda_{j'}^*)} = i (\lambda_j \lambda_{j'}^* (\xi_{j'} | \xi_j) + (\xi_{j'} | \mathcal{C} \xi_j)). \quad (37)$$

From this we can argue that, when the mode separation frequencies are less than the characteristic damping rate (assumed to be comparable for neighbouring modes), the contribution of off diagonal terms, as compared to diagonal terms in (32) is measured by the magnitude of an overlap expression between the displacements of the two modes that approaches zero in the limit of zero viscosity when $j \neq j'$. Provided such overlaps remain relatively small, it should be possible to neglect the off diagonal terms in the sum. This would be satisfied if the form of the eigenfunctions is preserved when the level of viscosity of interest is introduced. This is physically reasonable when the damping rate is small compared to the actual mode frequency and viscosity is important only in localised regions of the star/planet where the mode wavelength is small, that for example, could be modelled using a boundary layer

treatment, in such a way that the form of the normal modes of interest is preserved in most of the star/planet. Such a situation may occur for stars with radiative envelopes when g modes are effectively damped only near the surface. We shall assume it to be the case from now on so that (35) applies and inviscid modes can be used to evaluate the terms in the summation.

3.4 Angular momentum transfer rate and energy transfer rate as viewed in the inertial frame

Equation (35) gives the transfer rate of canonical energy defined in the rotating frame. The angular momentum transfer rate, \dot{L}_c , and the energy transfer rate as defined in the inertial frame, \dot{E}_I , follow by using the fact that for forcing with a harmonically varying potential with pattern speed $\omega_{m,k}/m$, the rates of change of angular momentum and canonical energy are related by $\dot{L}_c = m\dot{E}_c/\omega_{m,k}$. In addition, for a harmonically varying forcing potential, the rate of change of energy in the inertial frame is related to the rates of change of canonical energy and angular momentum through

$$\dot{E}_I = \dot{E}_c + \Omega \dot{L}_c = \dot{E}_c (\omega_{m,k} + m\Omega)/\omega_{m,k}. \quad (38)$$

Applying the above results separately to the contributions to the sum in equation (35) arising from distinct forcing frequencies, we obtain

$$\dot{L}_c = -4\pi \sum_{m,k,j} \frac{m\omega_{\nu,kj} |\lambda_j|^2 \mathcal{N}_j |S_{j,k}|^2}{\omega_{m,k} |N_j|^2 D_{k,j,j}}, \quad (39)$$

$$\dot{E}_I = -4\pi \sum_{m,k,j} \frac{\omega_{\nu,kj} |\lambda_j|^2 (\omega_{m,k} + m\Omega) \mathcal{N}_j |S_{j,k}|^2}{\omega_{m,k} |N_j|^2 D_{k,j,j}}. \quad (40)$$

Note that in the limiting case of a small stellar angular velocity directed perpendicular to the orbital plane, and under the assumption that the effect of rotation can be dealt with by simply Doppler shifting forcing frequencies, analogous equations have been derived by, e.g. Kumar et al. (1995).

3.5 A simplification in the case of dense spectrum of normal modes

The expressions (35) (39) and (40) can be significantly simplified in the case when the spectrum of eigenmodes contributing most significantly to the sums becomes dense with a typical distance between two neighbouring modes $|\omega_{j+1} - \omega_j|$ approaching a smooth function that we can identify with $|d\omega_j/dj| \ll |\omega_j|$ for $j \rightarrow \infty$. An example of this situation occurs for non rotating cool stars with radiative regions that support internal gravity modes. These have the required property in the high order/low frequency limit. In that case $|d\omega_j/dj| \rightarrow |\omega_j/j|$ for $j \rightarrow \infty$. In general, a high degree of regularity of the spectrum is required for it to have this property.

As above we suppose that the main contribution to the series comes from modes having near resonance with the external forcing frequencies. These are such that $\omega_j \approx \omega_{m,k}$ making the magnitude of the denominator, $D_{k,j,j}$, a small quantity for such modes. Let us assume that for a particular forcing frequency, $\omega_{m,k}$, the resonance condition is most closely satisfied for a particular mode with index $j = j_0$, (which we recall will vary with m and k). Accordingly we have

$$\omega_{j_0} = \omega_{m,k} + \Delta\omega_{j_0}, \quad (41)$$

where the offset, $\Delta\omega_{j_0}$, is such that $|\Delta\omega_{j_0}| < |d\omega_{j_0}/dj_0|$. We

approximate values of neighbouring mode frequencies by representing ω_j as a Taylor series in $j - j_0$ up to the linear term, thus $\omega_j \approx \omega_{j_0} + (d\omega_{j_0}/dj_0)l$, where $l = j - j_0$. This is then substituted in $D_{k,j,j}$ and the result used in equations (35) (39) and (40). In addition quantities other than $D_{k,j,j}$ in eq. (35) are assumed to depend smoothly on j . As the dominant contribution to the sums comes from values of j close to j_0 , such quantities can be evaluated for $j = j_0$ and $\omega_j = \omega_{m,k}$ and taken out of sums over j . In addition as the viscous decay rates are assumed small we set $\lambda_j = \omega_{m,k}$. Following the discussion in section 3.3.2, we assume the normal mode spectrum is approximately independent of dissipation so that the inviscid case may be adopted, we set N_j and \mathcal{N}_j equal to $\mathcal{N}_{j_0} \equiv N_{j_0}$.

In this way from (35) we obtain

$$\begin{aligned} \dot{E}_c &= -4\pi \sum_{m,k} \mathcal{A}_k \frac{\omega_{\nu,kj_0} \omega_{m,k}^2}{|d\omega_{j_0}/dj_0|^2} \frac{|S_{j_0,k}|^2}{N_{j_0}}, \text{ where} \\ \mathcal{A}_k &= \sum_{-\infty}^{+\infty} \frac{1}{(l + \delta)^2 + \kappa^2} \end{aligned} \quad (42)$$

and we formally extend the sum over the index (l) to $\pm\infty$,

$$\delta = \left| \frac{\Delta\omega_{j_0}}{d\omega_{j_0}/dj_0} \right|, \text{ and } \kappa = \left| \frac{\omega_{\nu,kj_0}}{d\omega_{j_0}/dj_0} \right|. \quad (43)$$

The factor \mathcal{A}_k can be evaluated by standard complex variable techniques to give

$$\mathcal{A}_k = \frac{\pi \sinh \pi \kappa \cosh \pi \kappa}{\kappa (\sinh^2 \pi \kappa + \sin^2 \pi \delta)}. \quad (44)$$

3.5.1 The case of moderately large viscosity or an averaged dissipation

Let us consider the case of large $\kappa \gg 1$ which occurs when the characteristic viscous decay rate of a resonant mode is much larger than the frequency difference between the relevant mode and a neighbouring mode. On the other hand the viscous decay time should also be much longer than the inverse mode frequency so that strong resonances with one particular mode are still possible.

As discussed by Goodman & Dickson (1998), physically this situation occurs when the decay time of a mode due to viscosity is much smaller than the time for wave propagation with the group velocity, associated with the potentially resonant modes, which is identified as $|d\omega_{j_0}/dj_0|^{-1}$. To see that the latter identification is reasonable, suppose that mode j is associated with an appropriately defined mean wavenumber k_j which, for large j , asymptotically approaches j/Δ , where Δ is a length scale that is characteristically of the order of the size of the region containing the modes. We have $|d\omega_{j_0}/dj_0| = |d\omega_j/dk_j|_{j=j_0}/\Delta$. It is seen that the right hand side of the above is the inverse of the time to propagate across Δ with the notional group velocity $|d\omega_j/dk_j|_{j=j_0}$.

In the limit $\kappa \gg 1$, $\mathcal{A}_k \approx \pi/\kappa$ and the expression for the energy transfer

$$\dot{E}_c = -4\pi^2 \sum_{m,k} \frac{\omega_{m,k}^2}{|d\omega_{j_0}/dj_0|} \frac{|S_{j_0,k}|^2}{N_{j_0}}. \quad (45)$$

² It is assumed hereafter that $|d\omega_{j_0}/dj_0|$ is not so small that the quadratic term in the Taylor expansion in l has to be retained.

does not depend on the value of viscosity. This is physically reasonable because propagating disturbances decay before they reach at least one boundary. Accordingly, the calculation of the tidal response can be performed in a subdomain of the star with boundary conditions on the state variables corresponding to outgoing waves where these leave the subdomain.

In Appendix B we support this conclusion by showing how to obtain (45) for a model of a non-rotating star, for which there is potential resonance with modes in a g mode spectrum, by a standard technique for solving the tidal response problem with appropriate radiation boundary conditions.

It is important to note that the expression (45) as well as the expressions (51) below apply not only to the case of moderately large viscosity, but they also give mean energy and angular momentum transfer rates obtained by averaging the factor \mathcal{A}_k over the interval $0 \leq \delta \leq 1$. Indeed, from equation (44) it is easy to see that $\int_0^1 d\delta \mathcal{A}_k = \pi/\kappa$. Thus, these expressions are formally valid when orbital or stellar parameters change at uniform rates such that a large range of δ is covered in the course of orbital and/or stellar evolution.

Let us stress, however, that when the energy dissipation is very weak the orbital evolution may become stalled at some fixed value of δ with the system located between resonances, see e.g. Terquem et al (1998).

3.5.2 The case of very low viscosity

Now let us consider the case of small $\kappa \ll 1$. In this case the factor $\mathcal{A}_k \approx 1/(\kappa^2 + \sin^2 \pi \delta/\pi^2)$, and we have from (42)

$$\dot{E}_c = -4\pi \sum_{m,k} \frac{\omega_{\nu,kj_0} \omega_{m,k}^2}{(\omega_{\nu,kj_0}^2 + (d\omega_{j_0}/dj_0)^2 \sin^2 \pi \delta/\pi^2)} \frac{|S_{j_0,k}|^2}{N_{j_0}}. \quad (46)$$

When δ is not very close to either zero or one the second term in the denominator of (46) is much larger than the first one. In this case the energy exchange is proportional to the value of viscosity as expected. However, close to resonances when $\omega_{\nu,kj_0} > |d\omega_{j_0}/dj_0| |\sin \pi \delta/\pi|$ we can neglect the second term. In this case the energy transfer is strongly amplified being inversely proportional to ω_{ν,kj_0} .

3.6 Energy and angular momentum transfers expressed in terms of overlap integrals

It is convenient to represent the quantities $S_{j,k}$ entering into expressions for the energy and angular momentum transfer and defined through equation (28) as an inner product of a quantity determined by the forcing potential and an eigenmode. Following the discussion above, the latter from now on will be taken to be those appropriate to the inviscid case $\nu = 0$ for which we recall $\boldsymbol{\eta}_j = \boldsymbol{\xi}_j$.

Using equation (8) for the Fourier components of the gradient of the forcing potential, we can write

$$\begin{aligned} S_{j,k} &= \frac{A_{m,k}}{2\pi} Q_j, \text{ with } Q_j = \left(\boldsymbol{\xi}_j | \int_0^{2\pi} d\phi e^{-im\phi} \nabla(r^2 Y_2^m) \right) \\ &= \int d^3 x \rho e^{-im\phi} \boldsymbol{\xi}_j^* \cdot \nabla(r^2 Y_2^m). \end{aligned} \quad (47)$$

The quantities Q_j , so defined, coincide with the so-called overlap integrals considered by e.g. Press & Teukolsky 1977, Ivanov & Papaloizou 2004, IP7 and others. Additionally, instead of using the

norm given by the expression (21) evaluated in the inviscid limit, it is more convenient to consider another norm

$$n_j = \pi((\xi_j|\xi_j) + (\xi_j|\mathcal{C}\xi_j)/\omega_j^2), \quad (48)$$

which reduces to a standard form in the limit of a non-rotating star. Let us express the expressions for the energy and angular momentum transfer in terms of Q_j and n_j . Substituting (47) and (48) into the expressions (35), (39) and (40) we obtain

$$\begin{aligned} \dot{E}_c &= - \sum_{m,k,j} \omega_{\nu,kj} \frac{|A_{m,k}\hat{Q}_j|^2}{D_{k,j,j}}, \\ \dot{L}_c &= - \sum_{m,k,j} m \frac{\omega_{\nu,kj}}{\omega_{m,k}} \frac{|A_{m,k}\hat{Q}_j|^2}{D_{k,j,j}} \quad \text{and} \\ \dot{E}_I &= - \sum_{m,k,j} \frac{\omega_{\nu,kj}(\omega_{m,k} + m\Omega)}{\omega_{m,k}} \frac{|A_{m,k}\hat{Q}_j|^2}{D_{k,j,j}}. \end{aligned} \quad (49)$$

where

$$\hat{Q}_j = Q_j / \sqrt{n_j}, \quad (50)$$

does not depend on normalisation of eigen functions, and we use equation (25) to obtain the last equality.

3.6.1 A dense spectrum of modes and moderately large viscosity

In this case equation (45) together with (38) leads to

$$\begin{aligned} \dot{E}_c &= -\pi \sum_{m,k} \frac{|A_{m,k}\hat{Q}_{j0}|^2}{|d\omega_{j0}/dj_0|}, \\ \dot{L}_c &= -\pi \sum_{m,k} \frac{m}{\omega_{m,k}} \frac{|A_{m,k}\hat{Q}_{j0}|^2}{|d\omega_{j0}/dj_0|}, \quad \text{and} \\ \dot{E}_I &= -\pi \sum_{m,k} \left(1 + \frac{m\Omega}{\omega_{m,k}}\right) \frac{|A_{m,k}\hat{Q}_{j0}|^2}{|d\omega_{j0}/dj_0|}. \end{aligned} \quad (51)$$

3.6.2 A dense spectrum of modes and very small viscosity

In this case equation (46) similarly leads to

$$\begin{aligned} \dot{E}_c &= - \sum_{m,k} \frac{\omega_{\nu,kj0}}{D_*} |A_{m,k}Q_{j0}|^2, \\ \dot{L}_c &= - \sum_{m,k} \frac{m\omega_{\nu,kj0}}{\omega_{m,k}D_*} |A_{m,k}Q_{j0}|^2, \quad \text{and} \\ \dot{E}_I &= - \sum_{m,k} \left(1 + \frac{m\Omega}{\omega_{m,k}}\right) \frac{\omega_{\nu,kj0}}{D_*} |A_{m,k}Q_{j0}|^2, \end{aligned} \quad (52)$$

where $D_* = (\omega_{\nu,kj0}^2 + (d\omega_{j0}/dj_0)^2 \sin^2 \pi\delta/\pi^2)$.

3.7 Energy and angular momentum transfer in the case of nearly parabolic orbit

The transfers of energy, ΔE_c , ΔE_I and angular momentum, ΔL_c , between the orbital motion and normal modes of the star during a periastron flyby on a formally parabolic orbit can also be expressed in terms of overlap integrals. For completeness we give expressions for these here. As mentioned above, in this case we should represent the results in terms of the Fourier transform of the tidal potential (see equation (2)) instead of the Fourier series used in the previous sections. Expressions for ΔE_c , ΔE_I and ΔL_c in terms of this were

derived by IP7³. As in the case discussed above they also can be represented as sums over quantities determined by the eigenmodes, which for a given m have eigenfrequency ω_k , that take the form

$$\begin{aligned} \Delta E_c &= 8\pi^3 \sum_{m,k} \frac{\omega_k^2 |S_m(\omega_k)|^2}{N_k}, \\ \Delta L_c &= 8\pi^3 \sum_{m,k} \frac{m\omega_k |S_m(\omega_k)|^2}{N_k} \\ \Delta E_I &= 8\pi^3 \sum_{m,k} \frac{\omega_k(\omega_k + m\Omega) |S_m(\omega_k)|^2}{N_k}. \end{aligned} \quad (53)$$

Representing again the quantities $S_m(\sigma)$ as a product of two factors determined by the orbit and by the eigenmodes, respectively, we obtain a relation corresponding to (47): $S_m(\sigma) = (1/2\pi)A_m(\sigma)Q_j$. Using the normalised overlap integrals defined in (50) we can rewrite (53) in a more convenient way as

$$\begin{aligned} \Delta E_c &= 2\pi^2 \sum_{m,k} |A_m(\omega_k)\hat{Q}_k|^2, \\ \Delta L_c &= 2\pi^2 \sum_{m,k} \frac{m}{\omega_k} |A_m(\omega_k)\hat{Q}_k|^2, \\ \Delta E_I &= 2\pi^2 \sum_{m,k} \left(1 + \frac{m\Omega}{\omega_k}\right) |A_m(\omega_k)\hat{Q}_k|^2. \end{aligned} \quad (54)$$

Note that the quantities $A_m(\sigma)$ entering (54) are discussed in e.g. Ivanov & Papaloizou (2011) for the general case of a parabolic orbit inclined with respect to equatorial plane of a rotating star⁴.

4 ROTATIONALLY MODIFIED GRAVITY MODES

4.1 Use of the traditional approximation

In what follows we are going to apply the general theory discussed above to the situation when the energy and angular momentum exchange between a rotating Sun-like star and its orbit occurs as a result of the excitation of internal rotationally modified gravity (g) modes. These are assumed to be associated with discreet low frequency spectrum.

For the cases of a periodic orbit and of a nearly parabolic flyby, the energy and angular momentum exchanges between the orbit and the star are characterised by two quantities associated with the normal modes. These are their eigenfrequencies ω_j and the corresponding overlap integrals Q_j . Thus for given orbital parameters and the rate of rotation of the star they determine the energy and angular momentum transfer rates in the case of 'moderately large viscosity' discussed above. In this regime the decay rate of a characteristically excited mode is small compared to its frequency as seen in the rotating frame. However, it is large compared to the

³ Note that in equation (5) of IP7 there should be a $(-)$ sign in front of the expression on the right hand side and in equation (28) the imaginary unit should be inside the summation.

⁴ Note that analogous expressions of Ivanov & Papaloizou 2011 differ by factor of two from those given in (54). This is due to the fact that in Ivanov & Papaloizou 2011, the summation over the azimuthal mode number is formally performed over positive and negative values of m , while in this paper we consider only positive azimuthal mode numbers. Note also that in this paper we take all quantities related to eigenmodes to be defined in the rotating frame, while in Ivanov & Papaloizou (2011) the inertial frame was used.

frequency separation of neighbouring modes. Then the waves associated with the rotationally modified (g) modes are dissipated near a boundary, preventing the setting up of standing waves. This leads to a radiation boundary condition for the forced problem. However, these disturbances are solutions of the adiabatic linearised problem away from localised dissipative regions. When dissipative effects are very weak, an alternative prescription for their action on stellar perturbations should be used (see sections 3.5.2 and 3.6.2).

The rotational angular frequency Ω , can be expressed in units of the natural frequency of the primary star, $\Omega_* = \sqrt{GM_*/R_*^3}$, where M_* and R_* are the stellar mass and radius, respectively. As the non spherical distortion of the star is $\propto (\Omega/\Omega_*)^2$ it may be assumed to be spherically symmetric when (Ω/Ω_*) is small with internal structure the same as in the non-rotating state.

The problem on hand has a symmetry property $\omega_j \rightarrow -\omega_j$, $\Omega \rightarrow -\Omega$. In the preceding sections it was assumed that Ω is positive while the eigenfrequencies ω_j can have either sign depending on direction of their propagation with respect to rotation of the star. In what follows it is convenient to adopt the alternative convention assuming that the eigenfrequencies are always positive while the case $\Omega > 0$ and $\Omega < 0$ correspond to the modes propagating in the prograde and retrograde sense with respect to rotation of the star, respectively.

When the orbital period is sufficiently large, tidal perturbations mainly interact modes of large radial order. In this case one can make use of the Cowling approximation which neglects variations of self-gravity when describing free normal modes of the star and the 'traditional' approximation, where only components of the Coriolis force perpendicular to the radius vector from the centre of the star are included. When the traditional approximation is adopted and the star is assumed to be spherical the equations describing the stellar pulsation admit separation of variables (see below). The problem of finding ω_j in such a setting has been considered by many authors, either by numerical means or using the WKBJ technique (e.g. Lee & Saio 1987, Berthomieu et al 1978 among others).

4.2 Calculation of the overlap integrals

However, less attention has been payed to the calculation of the overlap integrals, Q_j , Rocca (1987) attempted to calculate quantities called tidal resonant coefficients that are proportional to Q_j , following the earlier work by Zahn (1970) who studied the same quantities in the case of non-rotating star.

However, significant potential inaccuracies remain, for example, Berthomieu et al (1978) assumed that the WKBJ approach is approximately valid in the convective regions of the star bounding a radiative region where the waves associated with the modes propagate. Although they did not consider the overlap integrals, this is not appropriate for the modes of low azimuthal order that are important for the tidal problem. In addition, when calculating the eigenfrequencies and eigenmodes, it has been generally assumed that, in a radiative zone, the square of the Brunt - Väisälä frequency behaves approximately linearly with the distance to the base of convection zone. Since for some models solar mass stars, this approximation is not accurate (see Fig 4 below) we allow for a more general power law dependence⁵

⁵ Note that Provost & Berthomieu (1986) give WKBJ expressions for the displacement vector in a radiative region valid for our power law dependence. However, solutions in the convective are obtained numerically pre-

Our approach to calculation of Q_j is similar to the approach of Rocca (1987). However, in contrast to that work, we consider a general power law dependence, on the distance to the convection zone boundary, for the Brunt - Väisälä frequency. We revisit the calculation of ω_j and Q_j , mainly concentrating on calculation of Q_j , since calculation of the eigenfrequencies is relatively straightforward. We find two contributions to the overlap integrals, the first comes from the radiative region close to the base of convection zone and other comes from the convection zone itself. We correct some errors in earlier work, and find the asymptotic form at low forcing frequency. We compare the results we obtain with numerical calculations for solar mass models in a regime where these can be performed.

4.3 Linearised equations for normal modes

Using the Cowling and traditional approximation in (5), while neglecting dissipative effects and the external forcing potential, we get

$$-\omega^2 \xi - 2i\omega (\Omega \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}} \times \xi + \mathcal{C} \xi = 0 \quad (55)$$

where

$$\mathcal{C} \xi = \frac{1}{\rho} \nabla P' - \frac{P'}{\rho \Gamma_1 P} \nabla P + N^2 (\xi \cdot \hat{\mathbf{r}}) \hat{\mathbf{r}}, \quad (56)$$

$$P'(\xi) = -\Gamma_1 P \nabla \cdot (\xi) - \xi \cdot \nabla P \quad (57)$$

and the square of the Brunt - Väisälä frequency is given by

$$N^2 = \frac{1}{\rho} \frac{dP}{dr} \left(\frac{1}{\rho} \frac{d\rho}{dr} - \frac{1}{\Gamma_1 P} \frac{dP}{dr} \right), \quad (58)$$

where $\Gamma_1 = (\partial \ln P / \partial \ln \rho)_{ad}$, with the subscript *ad* indicating that the derivative is taken at constant entropy.

We adopt spherical polar coordinates (r, θ, ϕ) with ξ_r , ξ_θ and ξ_ϕ being the components of the vector ξ in these coordinates, the component of Ω acting in the radial direction is $\Omega \cdot \hat{\mathbf{r}} \equiv (\Omega \cos \theta, 0, 0)$ where $\hat{\mathbf{r}}$ is the unit vector in the radial direction. It will be taken as read that all quantities of interest will be taken to be expressed in terms of Fourier series as specified as in (1) with the equations applying to separate coefficients. Here ρ is the density and P is the pressure in the background state. The gravitational acceleration at radius r is $g = GM(r)/r^2$ and $M(r)$ is the mass enclosed within a sphere of radius r .

The density and pressure perturbations are ρ' and P' respectively. We note that ρ' , P' and ξ_r satisfy the condition for adiabatic perturbations which gives

$$P' = c_s^2 \rho' - \frac{c_s^2 \rho N^2}{g} \xi_r, \quad (59)$$

where $c_s = \sqrt{\Gamma_1 P / \rho}$ is the adiabatic sound speed. Note that if $N^2 \geq 0$ everywhere as assumed in this paper, the operator \mathcal{C} is positive and the linear eigenvalue problem can be brought into the self-adjoint form (17). For simplicity we suppress the mode index (j) below, wherever this doesn't lead to any ambiguity.

As is well known (e.g. Unno et al 1989), both the forced response problem and the unforced ($\Psi = 0$) problem considered in this section can be solved by adopting the separation of variables anzatz:

cluding the development of an analytic asymptotic representation of the overlap integrals, which were not considered, at low forcing frequency.

$$\xi_r = \frac{\xi(r)H(\theta)}{\sqrt{2\pi}}, \rho' = \frac{\hat{\rho}(r)H(\theta)}{\sqrt{2\pi}}, \text{ and } P' = \frac{\hat{P}(r)H(\theta)}{\sqrt{2\pi}} \quad (60)$$

together with

$$\begin{aligned} \xi_\phi &= \frac{1}{\sqrt{2\pi}} \frac{\xi^S(r)}{(1 - \nu^2 \cos^2 \theta)} \left(\frac{im}{\sin \theta} H(\theta) - i\nu \cos \theta \frac{dH(\theta)}{d\theta} \right), \\ \xi_\theta &= \frac{1}{\sqrt{2\pi}} \frac{\xi^S(r)}{(1 - \nu^2 \cos^2 \theta)} \left(\frac{dH(\theta)}{d\theta} - m\nu \cot \theta H(\theta) \right). \end{aligned} \quad (61)$$

Here quantities with a *hat* are functions only of r , $\nu = 2\Omega/\omega$ and the factor $1/\sqrt{2\pi}$ is inserted for convenience (see below). We remark that the separation of the ϕ dependence through the usual factor $\exp(im\phi)$ is here taken as read.

Substituting equations (60) and (61) into equations (55) - (59) we verify separability provided that the function $H(\theta)$ is determined through solving the eigenvalue problem specified by

$$\hat{L}H = -\Lambda H, \text{ where the operator } \hat{L} \text{ is defined by} \quad (62)$$

$$\begin{aligned} \hat{L} &= \frac{d}{d\mu} \left(\frac{(1 - \mu^2)}{(1 - \nu^2 \mu^2)} \frac{d}{d\mu} \right) \\ &+ \frac{1}{(1 - \nu^2 \mu^2)} \left(\frac{m\nu(1 + \nu^2 \mu^2)}{(1 - \nu^2 \mu^2)} - \frac{m^2}{(1 - \mu^2)} \right), \end{aligned} \quad (63)$$

with $\mu = \cos \theta$. The eigenvalues, Λ , specified by (62), which defines the Laplace tidal equation, are also separation constants. The associated eigenfunctions are Hough functions (see e.g. Longuet-Higgins, 1968, for a discussion of the eigenvalues and eigenfunctions).

After separating the variables by the means as indicated above, it is found that the angular components of (55) require that

$$\xi^S = \frac{\hat{P}}{r\omega^2 \rho}. \quad (64)$$

The radial component of (55) together with (57) then lead to two ordinary differential equations for the radial dependence of $\xi(r)$ and \hat{P} . These equations take the form

$$\frac{d\xi}{dr} = -\left(\frac{2}{r} + \frac{1}{\Gamma_1 P} \frac{dP}{dr}\right) \xi + \frac{\hat{P}}{\rho} \left(\frac{\Lambda}{\omega^2 r^2} - \frac{1}{c_s^2}\right), \quad (65)$$

and

$$\frac{d\hat{P}}{dr} = \rho(\omega^2 - N^2)\xi + \frac{\hat{P}}{\Gamma_1 P} \frac{dP}{dr}. \quad (66)$$

In general Λ is a function of ν , and, accordingly, ω . Therefore, when $\Omega \neq 0$ the eigenvalues Λ and ω defined through the solution of equations (65), (66) and (62) must be determined together.

In the degenerate case $\Omega = 0$, $\nu = 0$, (62) reduces to Legendre's equation. Its solutions are accordingly are associated Legendre functions $P_l^m(\mu)$ and $\Lambda = l(l+1)$, where l is an natural number. Since in this limit the tidal problem is such that only eigenmodes corresponding to $l = 2$ are coupled to the dominant quadrupole component of the tidal potential, Λ can be set to 6. The eigenvalue problem for the determination of ω is accordingly decoupled from the angular one in the non-rotating case. Equations (65)-(66) then coincide with standard equations describing the free modes of pulsation of a non-rotating star under the Cowling approximation, see e.g. Christensen-Dalsgaard (1998).

4.4 Properties of the Laplace tidal equation

We here summarize properties of equation (62) required for our purposes. When the parameters ν and m are fixed the operator \hat{L} is

self adjoint with respect to the natural inner product defined by integration over $(-1, 1)$ with respect to μ . There is a countable infinite set of solutions corresponding to different characteristic values of Λ , Λ_i . The corresponding eigenfunctions, H_i , are orthogonal and normalised such that

$$\int_{-1}^1 d\mu H_i(\mu) H_k(\mu) = \delta_{ik}. \quad (67)$$

In general, the eigenvalues, Λ_i , can be either positive or negative. In this paper we assume below that they are non-negative, which is necessary for the associated solutions to correspond to wave-like perturbations in stably stratified regions, as required for potentially resonant rotationally modified gravity modes. Thus the Λ_i , $i = 1, 2, \dots$, will be arranged such that $\Lambda_{i+1} > \Lambda_i$ with Λ_1 corresponding to the minimum non-negative eigenvalue for a given ν and m .

As stated above, when $\nu = 0$, the H_i are proportional to associated Legendre functions, P_l^m . In the general case, $\nu \neq 0$, the functions H_i may be represented as an infinite series of associated Legendre functions through

$$H_i = \sum_{n \geq 0} \alpha_{i,n} \tilde{P}_{(2n+2)}^m, \text{ with } \tilde{P}_{(2n+2)}^m = \sum_{i>0} \alpha_{i,n} H_i. \quad (68)$$

Here the scaled associated Legendre functions

$$\tilde{P}_l^m(\mu) = \sqrt{\frac{(2l+1)(l-m)!}{2(l+m)!}} P_l^m(\mu) \quad (69)$$

satisfy the condition $\int_{-1}^1 d\mu \tilde{P}_l^m(\mu) \tilde{P}_{l'}^m(\mu) = \delta_{l,l'}$. The second equality in (68) follows from orthogonality and normalisation of each of the sets of functions H_i and $\tilde{P}_{(2n+2)}^m$. We calculate the decomposition of the tidal potential as a series of Hough functions using equation (68) truncating the summation at sufficiently large value of, n , with help of the procedure described in Ogilvie & Lin (2004).

As the dominant tidal potential has the angular dependence of a quadrupole, it follows from equation (47), that the overlap integrals involve a product of \tilde{P}_2^m and H_i , and therefore, the coefficients

$$\alpha_i(\nu) \equiv \alpha_{i,0} = \int_{-1}^1 d\mu H_i \tilde{P}_2^m \quad (70)$$

play an important role. It describes the 'angular' coupling of the rotationally modified g-modes with the tidal field within the framework of the traditional approximation.

For moderate values of ν (for which, $|\nu| \leq 3$, say) only the quantities corresponding to $i = 1$ are important. We calculate $\Lambda_1(\nu)$ and $\alpha_1(\nu)$ numerically as well as an additional quantity, $I_\theta(\nu)$, determined by the Hough functions and defined below (see equation (84) and plot them in Fig. 1 for $m = 2$ and in Fig. 2 for $m=0$). Note that in the latter case from the definition of $\Lambda_1(\nu)$ and $\alpha_1(\nu)$ it follows that they are even functions of ν while I_θ is an odd one. Therefore, we show only positive values of ν in Fig. 2.

The case $m = 2$ is the most important for tidal interaction. We see from Fig. 1 that $\Lambda_1(\nu)$ and $-I_\theta(\nu)$ increase as ν decreases. On the other hand $\alpha_1(\nu)$ increases with ν until $\nu = -1$, levelling off at larger values. As $\alpha_1(\nu)$ measures the strength of coupling to the tidal field, this latter behaviour results in the stronger excitation of modes associated with positive values of ν .

The values of ν are in general different for different modes since these are obtained from the simultaneous solution of eigenvalue problems in the radial and θ directions. Nonetheless, we clas-

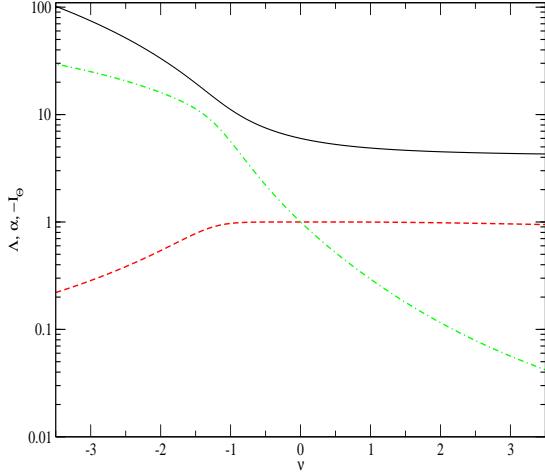


Figure 1. The values of Λ_1 (solid line), α_1 (dashed line) and $-I_\theta$, where I_θ is defined through equation (84) for $m = 2$ as functions of the parameter ν .

sify the eigenmodes using two indices, the index (i) corresponding to a particular $\Lambda_i(\nu_i)$, which accordingly indicates the order of the angular eigenvalue, and the index (n) describing the number of nodes in the radial direction, for a given m . Thus our mode index j , used in previous sections, should be understood as a pair (i, n) .

Note that when the star is non-rotating, and, accordingly, $\nu = 0$, from equation(61) it follows that $\xi = \xi^m Y_l^m \mathbf{e}_r + \xi^S r \nabla Y_l^m$, where \mathbf{e}_r is the unit vector in the radial direction.

4.5 An expression for the overlap Q_j in the low frequency limit in the traditional approximation

We now reduce the expression (47) for the overlap integral Q_j to a form involving an integral over the radial coordinate only in the traditional approximation. This is useful for either performing numerical calculations or undertaking further asymptotic analysis (see below). We work on the expression for Q_j given by equation (47).

It is convenient to express the term $\nabla(r^2 Y_2^m)$ that appears there as $2r Y_2^m \mathbf{e}_r + r^2 \nabla Y_2^m$. We then split the integral for Q_j into two parts, the first containing $2r Y_2^m \xi_j^* \cdot \hat{\mathbf{r}}$ and the second containing $r^2 \xi_j^* \cdot \nabla Y_2^m$. We remark that the forms of both the free and forced problems specified by (55) are such that after separation of variables, the functions ξ and ξ^S written as ξ_j and ξ_j^S , when these correspond to a free normal mode j , may be taken to be real. Noting that $Y_2^m = \tilde{P}_2^m \exp(im\phi)/\sqrt{2\pi}$ and using (70), the first part of (47) can be transformed to become

$$2\alpha_i \int_0^{R_s} r^3 dr \rho \xi_j. \quad (71)$$

To deal with the second part we first note that equation (63) is equivalent to the relation

$$r \nabla_\perp (e^{im\phi} \xi_j) = -\Lambda_i \xi_j^S H_i e^{im\phi} / \sqrt{2\pi}, \quad (72)$$

where $\nabla_\perp \equiv \nabla - \hat{\mathbf{r}} \hat{\mathbf{r}} \cdot \nabla$. Making use of (72), performing an

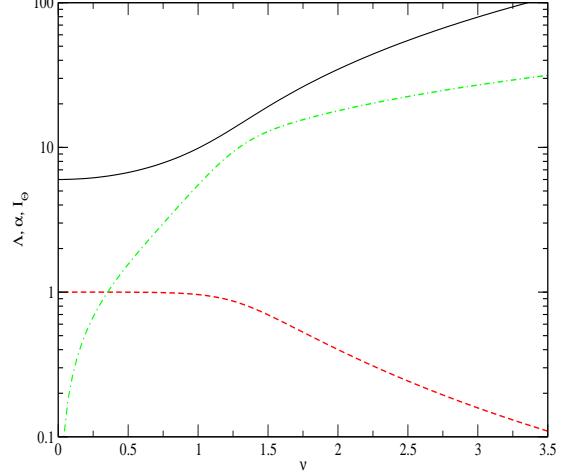


Figure 2. Same as Fig. 1 but the case $m = 0$ is shown and the dot-dashed line shows I_θ .

integration by parts, and noting (68) and (70), the second part can be reduced to the form

$$\alpha_i \Lambda_i \int_0^{R_s} r^3 dr \rho \xi_j^S. \quad (73)$$

Summing the contributions of the two parts, we obtain

$$Q_j = \alpha_i Q_{rj}, \text{ with } Q_{rj} = \int_0^{R_s} r^3 dr \rho (2\xi_j + \Lambda_i \xi_j^S). \quad (74)$$

When the star is non-rotating $\alpha_i = 1$, $\Lambda = 6$, and the expression (74) reduces to its standard form, see e.g. PT, IP4.

We remark that, when considering which regions of the star contribute to the overlap integrals, the original expression (47) can also, after a straightforward integration by parts, be reduced to an integral for which the integrand is proportional to the mode density perturbation. Viewed in this way, only regions with non vanishing density perturbation contribute, consistent with expectation for a tidal interaction controlled by the gravitational potential due to a perturber.

The expression for Q_{rj} can be transformed into a form that is explicitly $\propto \omega_j^2$ which is thus very convenient in the low frequency limit, see also Rocca (1987). In order to proceed we first use (64) to express \hat{P} in terms of ξ^S and substitute the result in equations (65) and (66). These then yield the two relations:

$$\xi_j = \frac{\omega_j^2}{N^2} \left(\xi_j + \frac{rN^2}{g} \xi_j^S - (r\xi_j^S)' \right), \quad (75)$$

$$\xi_j^S = \frac{1}{\Lambda_i} \left(\frac{1}{\rho r} (r^2 \rho \xi_j)' + \frac{rN^2}{g} \xi_j + \frac{\omega_j^2 r^2}{c_s^2} \xi_j^S \right), \quad (76)$$

where a prime indicates differentiation with respect to r . Now we substitute the right hand side of equation(76) for ξ_j^S in (74) and then integrate the term proportional to $(r^2 \rho \xi_j)'$ by parts, assuming that ξ_j is smooth, but we allow for the density ρ to undergo a discontinuous jump at some point $r = r_*$ as is found to occur in some stellar

models. In addition we assume that there are no boundary contributions coming from the surface or centre of the star. We accordingly obtain

$$Q_{rj} = \int_0^{R_*} r^4 dr \rho \left(\frac{N^2}{g} \xi_j + \frac{\omega_j^2 r}{c_s^2} \xi_j^S \right) - r^4 \xi_j [\rho]^\pm, \quad (77)$$

where the term in square brackets in (77) should be understood as being equal to the difference between the expression in square brackets taken at radii $r_* + \epsilon$ and $r_* - \epsilon$ and then taking the limit $\epsilon \rightarrow 0$.

In order to get an expression for Q_r that is explicitly proportional to ω_j^2 we substitute the right hand side of equation (75) for ξ_j into (77) and then integrate the term proportional to $(r\xi_j^S)'$ by parts assuming the same conditions as above. We then get

$$Q_{rj} = \omega_j^2 \int_0^{R_*} r^4 dr \rho \left(\frac{\xi_j}{g} + \frac{\xi_j^S}{r^3} \left(\frac{r^4}{g} \right)' \right) - S^\pm \quad (78)$$

where the surface term

$$S^\pm = r^4 \xi_j [\rho]^\pm - \omega_j^2 \frac{r^5}{g} [\rho \xi_j^S]^\pm. \quad (79)$$

In order to evaluate S^\pm we take equation (75), multiply both sides by ρN^2 and substitute the expression given by (58) for the square of the Brunt - Väisälä frequency. Integrating the result between $r_* + \epsilon$ and $r_* - \epsilon$, making use of the fact that the displacements are finite at $r = r_*$ and that the pressure gradient is equal to $-\rho g$, we readily obtain $S^\pm = 0$.

So far the expressions for Q_{rj} , though adapted to low frequencies, are exact and can be used for numerical evaluation. To facilitate analytic discussion in the relevant limit of low frequency, we express the integrand in (77) in terms of ξ_j only, and in so doing, obtain a simplified expression for Q_{rj} that neglect terms of order ω_j^4 and higher.

In order to do this, we substitute the right hand side of equation (76) into (78), first noting that the last term may be neglected to the order we are working. Integrating again by parts, assuming no boundary contributions or internal jumps in ξ_j^S , we obtain

$$\begin{aligned} Q_{rj} &= \omega_j^2 \int_0^{R_*} r^4 dr \rho h(r) \xi_j + O(\omega_j^4), \text{ where} \\ h(r) &= \frac{1}{g} \left(1 - \frac{(12 - 4v)}{\Lambda_i} \right) \\ &+ \frac{1}{\Lambda_i} \left(\frac{r^2 g''}{g^2} - \frac{2(rg')^2}{g^3} + \frac{(8 - v)rg'}{g^2} \right). \end{aligned} \quad (80)$$

Here $v = N^2 r / g$.

4.6 An expression for the norm

In order to calculate energy and angular momentum exchange rates resulting from tidal interaction, in addition to the overlap integrals Q_j , we also require the norms n_j (see eg. section 3.6). Here we develop a form for n_j in terms of one dimensional angular and radial integrals that can be readily evaluated. To do this we transform the expression (48) for the norm, n_j , so that it involves integrals only over the radial coordinate. Using the that ξ_j satisfies, $\omega^2 \xi_j = \mathcal{C} \xi_j + \omega \mathcal{B} \xi_j$, (48) can also be written as

$$n_j = \frac{2\pi}{\omega_j^2} ((\xi_j | \mathcal{C} \xi_j) + \frac{\omega_j}{2} (\xi_j | \mathcal{B} \xi_j)). \quad (81)$$

Using equation (56) to specify (81) in terms of the linear perturbations, making use of (57)-(66), performing an integration over the angular coordinates by parts, and then making use of (72), we obtain

$$(\xi_j | \mathcal{C} \xi_j) = \frac{\omega_j^2}{2\pi} \int_0^{R_*} r^2 dr \rho (\xi_j^2 + \Lambda_i (\xi_j^S)^2). \quad (82)$$

This expression reduces to the standard one in the non-rotating case when $\Lambda = 6$. Using (12), with Ω replaced by its radial component $(\Omega \cdot \hat{r}) \hat{r}$ as appropriate for the traditional approximation, together with (61), we obtain

$$(\xi_j | \mathcal{B} \xi_j) = \frac{2}{\pi} \Omega I_r I_\theta, \quad (83)$$

where

$$\begin{aligned} I_r &= \int_0^{R_*} r^2 dr \rho (\xi_j^S)^2, \quad \text{and} \quad I_\theta = \int_{-1}^1 \frac{\mu d\mu}{(1 - \nu^2 \mu^2)^2} \times \\ &\left[\nu \mu (1 - \mu^2) \left[\frac{dH}{d\mu} \right]^2 + m(1 + \nu^2 \mu^2) H \frac{dH}{d\mu} + \frac{m^2 \nu \mu H^2}{1 - \mu^2} \right]. \end{aligned} \quad (84)$$

Substituting (82) and (83) in (81) we get $n_j = n_{st} + n_r$,

$$\text{where } n_{st} = \int_0^{R_*} r^2 dr \rho (\xi_j^2 + \Lambda_i (\xi_j^S)^2), \text{ and } n_r = \nu I_r I_\theta. \quad (85)$$

We remark that the integrals with respect to r can be straightforwardly evaluated once the eigenfunctions are known. The angular integral I_θ is a function of ν only. The dependence of $-I_\theta$ on ν is illustrated in Fig. 1.

5 NUMERICAL CALCULATION OF ω_j , Q_j AND n_j FOR ROTATIONALLY MODIFIED GRAVITY MODES FOR MODELS OF SOLAR-TYPE STARS

In this section we describe the numerical evaluation of the eigenfrequencies ω_j and the overlap integrals Q_j for rotationally modified g -modes under the traditional approximation for stars with solar-type structure. We also highlight aspects of the models that are relevant to a subsequent asymptotic analysis,

5.1 Stellar models considered

Our calculations are for a rotating star with a radiative core and extended convective envelope. The structure is assumed not to be modified by rotation, such modifications being a second order effect. This has the advantage that standard spherical models can be used.

We have considered an almost zero age model of the star with $M_* = M_\odot \approx 2 \cdot 10^{33} g$ and age $t \approx 1.66 \cdot 10^8 yr$ provided to us by I.W. Roxburgh (see Roxburgh 2008 for a description of his numerical code) referred hereafter to as IR model, and also a model of the present day Sun (Christensen-Dalsgaard et al. 1996) with age $t = 4.6 \cdot 10^9 yr$, referred hereafter to as the CD model. Plots the density ρ and the square of the Brunt - Väisälä frequency, N^2 , against radius, for these models are given in Fig. 3. Note that the total stellar radius of the IR model, R_* is smaller than that of the present day Sun, taken to be $R_* = R_\odot = 7 \cdot 10^{10} cm$. For the IR model we have $R_* = 6.3 \cdot 10^{10} cm$ approximately.

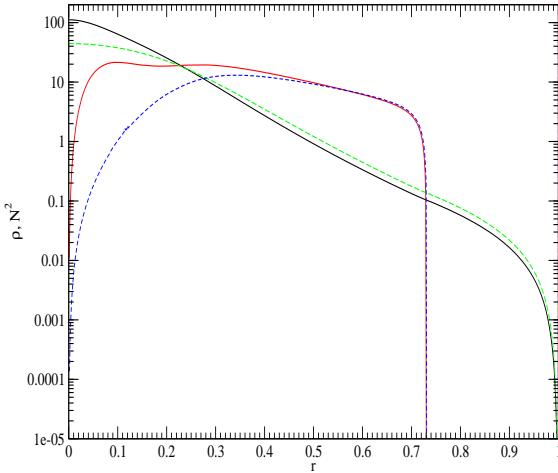


Figure 3. The density ρ (in units of the mean density $3M_*/(4\pi R_*^3)$) and square of the Brunt - Väisälä frequency, N^2 (in units of GM_*/R_*^3) as functions of the radius r expressed in units of R_* . The solid curves correspond to the CD model while the dashed ones are for the IR model. The curves monotonically decreasing with r are for the density distributions, while the curves having maxima at some values of r are for the Brunt - Väisälä frequencies.

5.2 Numerical approach

The eigenfrequencies are obtained by finding appropriate solutions of (65) and (66), in the core and in the envelope and applying matching conditions at the base of convection zone. Since we consider low frequency modes we can neglect the last term in the brackets multiplying \hat{P} in (66) as this is important only very close to the surface in the limit $\omega^2 \rightarrow 0$ of interest. The radial eigenvalue problem is solved together with the azimuthal eigenvalue problem. To solve the former the standard fourth order Runge-Kutta method was used. To solve the latter for a given eigenfrequency, we adopted the matrix method of Ogilvie and Lin 2004. We employed $\sim 10^6 - 10^7$ grid points in the radial direction in order to ensure numerical convergence for eigenfrequencies $\omega < \sim 0.1$ in natural units. Since the CD and IR models were provided with a much smaller number of grid points, 1282 and 2001 respectively, linear interpolation was employed to supply the data for our finer grid. Having obtained the eigenfunctions, we then use the expressions (78) and (84) to calculate the overlap integrals and norms.

5.3 Asymptotic analysis in the limit $\omega \rightarrow 0$.

In subsequent sections we shall discuss the asymptotic form of the normal modes, overlap integrals and norms in the low frequency limit following a WKBJ approach. In order to proceed we first discuss the behaviour of N^2 in the vicinity of the interface between the radiative and convective regions of the stellar model, this being an important consideration for the asymptotic analysis.

Model	A	q	r_c
CD1	41.54	0.747	0.722
CD2	24.32	0.617	0.721
IR1	41.4	0.737	0.731
IR2	23.1	0.564	0.73
CDL	106	1	0.724
IRL	110	1	0.733

Table 1. Results of fitting of the dependence of N^2 on r for the CD and IR models. Different entries in the table correspond to either general fits made with different data sets, or fits assuming $q = 1$, as described in the text (see also Fig. 4). Note that A and r_c are given in natural units such that r and N^2 are expressed in the units of R_* and the natural stellar frequency GM_*/R_*^3 , respectively.

5.4 Behaviour of the Brunt - Väisälä frequency near the base of the convection zone

In previous studies (e.g. Zahn 1970, Berthomieu et al 1978, Rocca 1987) it was assumed that near the boundary between convective and radiative zones, but on the radiative side, $N^2 \propto$ the difference $x = |r_c - r| \ll r$, where r_c is the radius of the boundary, defined by the condition $N^2(r_c) = 0$. However, this is not to be the case for many stellar models (see Barker 2011). In particular, as seen from Fig. 4, for the solar models we use, N^2 decreases more slowly with x than expected for a linear dependence. Here we consider more general possibilities, by assuming that when $r_c > r$, so that $x = r_c - r > 0$, for $x \ll r$ we may write

$$N^2 \approx Ax^q, \quad (86)$$

where q is a constant, see also Provost & Berthomieu (1984). Note that as is seen from Fig. 4 the dependence of N^2 on $x = r_c - r$ is not strictly given by a single power law. Local fits result in smaller values of q at larger values of x . However, this feature is not very important for our purposes. For our estimates of the overlap integrals below, only the local value of q that is found to be applicable to the range of x such that $1 > N^2 > 0.05$ is important. This is needed to find the contribution to Q_r given by equation (112), which plays a role only when eigenfrequencies are in the range $\omega \sim 0.2 - 1$, see Figs. 7-9.

We use equation (86) to fit the dependence of the Brunt - Väisälä frequency on r in the CD and IR models in the vicinity of the base of convection zone using the Levenberg-Marquardt algorithm.

Since the dependence is not, in fact, an exact power law we made fits using two different data sets in order to explore the range of the resulting values of A and q . For the first data set the number of data points is constrained approximately by the condition $N^2 < 2$, while for the second, and more extended data set, the number of points is constrained by the condition $N^2 < 4$. Additionally, we made fits to the data fixing the power index $q = 1$ for the data sets with $N^2 < 2$. The results are illustrated in Fig. 4 and tabulated in table 1.

The results labelled CD1, CD2 and CDL are respectively for the fits using the data sets with $N^2 < 2$, $N^2 < 4$, and with assumed linear dependence of N^2 on x , for the CD model of the star. The labels IR1, IR2 and IRL describe the same type of fits made for the IR model.

From Fig. 4 we see that the fits obtained with a smaller number of data points give somewhat larger value of $q \approx 0.74$ whereas the larger data sets are fitted quite well by (86) with $q \approx 0.6$. This indicates that the value q required for the most accurate fit may decrease with the radial extent of the region of interest. This in turn

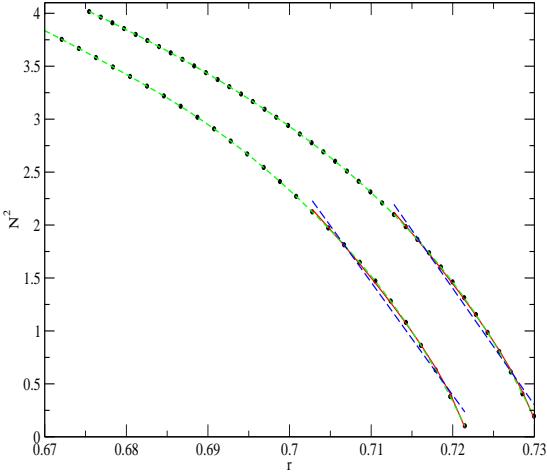


Figure 4. N^2 close to the base of convection zone plotted as a function of radius, r , expressed in units of R_* . The filled circles show the data points used to make the fits. Those with larger values of N^2 at a given r correspond to the IR model. The solid and dashed curves show the CD1, IR1, and the CD2, IR2 fits respectively. The dashed straight lines show the fits CDL and IRL which had $q = 1$, see table 1 for the values of the parameters A and q appropriate to these fits.

may be a function of frequency when normal modes are considered. We discuss below how quantities obtained in our analysis depend on the assumed fit.

6 CALCULATION OF EIGENFREQUENCIES, EIGENFUNCTIONS, OVERLAP INTEGRALS AND NORMS FOR HIGH ORDER ROTATIONALLY MODIFIED GRAVITY MODES FOR MODELS OF SOLAR-TYPE STARS FOLLOWING A WKB APPROACH

In order to provide a check on numerical calculations and to extend results to frequencies below which the numerical approach can run into difficulties on account of inadequate resolution, we perform an asymptotic analysis following a WKB approach. We first obtain expressions for eigenfrequencies and eigenmodes and then go on to use them to evaluate overlap integrals and norms.

6.1 Asymptotic solution of the radial eigenvalue problem providing expressions for the radial eigenfunctions and associated eigenfrequencies

We now discuss the asymptotic form of the eigenfrequencies and eigenfunctions found from solving the radial eigenvalue problem in the low frequency limit adopting a WKB approach. In what follows, for simplicity we make the assumption that $\Gamma_1 = \text{const}$ throughout the regions of the star of interest, setting $\Gamma_1 = 5/3$ in numerical expressions. Note that these assumptions can be easily relaxed if needed. Similarly the term involving $1/c_s^2$ in (65) becomes important only very close to the surface it be neglected in

the same limit. This term can also be taken into account in a more advanced variant of our scheme.

6.1.1 The WKB solution in the radiative core

Under the assumptions mentioned above equations (65) and (66) can be combined to yield the second order differential equation

$$\frac{P^{1/\Gamma_1}}{\rho} \frac{d}{dr} \left(\frac{\rho}{P^{2/\Gamma_1}} \frac{d}{dr} (r^2 P^{1/\Gamma_1} \xi) \right) + \Lambda \left(\frac{N^2}{\omega^2} - 1 \right) \xi = 0. \quad (87)$$

An approximate WKB solution in the radiative core, sufficiently far from the centre that $N^2/\omega^2 - 1$ may be replaced by N^2/ω^2 in (87), has the standard form (see e.g. Christensen-Dalsgaard 1998)

$$\begin{aligned} \xi &= C_{WKB} r^{-3/2} (\rho N)^{-1/2} \sin(W/\omega + \phi_{WKB}), \text{ where} \\ W &= \Lambda^{1/2} \int_0^r \frac{dr}{r} N, \end{aligned} \quad (88)$$

C_{WKB} is a constant and the angle ϕ_{WKB} is fixed by the condition that the solution matches onto a regular one as $r \rightarrow 0$ as

$$\phi_{WKB} = -\frac{l\pi}{2}, \text{ with } l = \frac{1}{2}(\sqrt{1+4\Lambda}-1), \quad (89)$$

see e.g. Vandakurov (1968)⁶. Note that l is not, in general, an integer. When the star is non-rotating we recall that $\Lambda = 6$, and, accordingly, $l = 2$.

6.1.2 The solution close to the base of convective envelope

We look for a solution that is valid in a radial region of extent $r \ll r_c$ in the radiative region close to its boundary with the convection envelope. To do this we substitute (86) in (87), neglecting unity in the brackets in the last term, the variation of P and ρ , and recalling that $x = r_c - r$. Equation (87) then reduces to

$$\frac{d^2 \xi}{dx^2} + \frac{\Lambda A x^q}{r_c^2 \omega^2} \xi = 0. \quad (90)$$

The general solution of (90) is a linear combination of Bessel function $J_\nu(z)$ such that

$$\begin{aligned} \xi &= \sqrt{x} (C_1 J_{\frac{1}{q+2}}(z) + C_2 J_{-\frac{1}{q+2}}(z)), \text{ where} \\ z &= \frac{K}{\omega} x^{\frac{q+2}{2}}, \quad K = \frac{2\sqrt{\Lambda A}}{(q+2)r_c}. \end{aligned} \quad (91)$$

Note that the expression (91) is equivalent to what is obtained in Provost & Berthomieu (1984) in the limit of small x provided that one discards higher order terms as discussed in their paper.

The solutions (88) and (91) must be matched in a region where $x/r \ll 1$ but $z \gg 1$. This is easily accomplished by the standard method of comparing the former with the latter after utilising the asymptotic expansion of the Bessel functions for large z . The matching procedure allows us to express the constants C_1 and C_2 in (91) in terms of C_{WKB} through

$$\begin{aligned} C_1 &= -\sqrt{\frac{\pi K}{2\omega}} C_* \frac{\cos(\phi_c - \pi q/(4(2+q)))}{\sin(\pi/(2+q))}, \\ C_2 &= \sqrt{\frac{\pi K}{2\omega}} C_* \frac{\cos(\phi_c - \pi(4+q)/(4(2+q)))}{\sin(\pi/(2+q))}, \end{aligned} \quad (92)$$

⁶ It can be shown that close to the centre we have approximately: $\xi_{\text{core}} \propto r^{-3/2} J_{l+1/2}(C_{\text{core}} r)$, where $J_\nu(z)$ is a Bessel function and C_{core} is a constant. This solution is regular when $l \geq 1$. At sufficiently large r the solution should be matched to (88). This determines the phase ϕ_{WKB} .

where $C_* = C_{WKBJ} \rho_c^{-1/2} r_c^{-3/2} A^{-1/4}$, and

$$\phi_c = \phi_{WKBJ} + \frac{\sqrt{\Lambda}}{\omega} \int_0^{r_c} \frac{dr}{r} N. \quad (93)$$

But note that the approximations that lead to equation (90) imply that this equation must be modified when $N^2 \leq \omega^2$, which corresponds to

$$x < x_{crit} = A^{-\frac{1}{q}} \omega^{\frac{2}{q}}. \quad (94)$$

On the other hand from (91) it follows that when $z \ll 1$ the solution approximately satisfies $d^2\xi/dx^2 = 0$ and the neglected term cannot produce significant changes over the small shrinking domain $(0, x_{crit})$ as $\omega \rightarrow 0$.

6.1.3 The solution in the convective envelope

Extending the solution into the convective region, we can set approximately $N^2 = 0$. The stellar gas, therefore, can be approximated as having barotropic equation of state with $P = K_c \rho^{\Gamma_1}$, which corresponds to a polytrope with polytropic index equal to 1.5. Assuming that the convective envelope has a negligible mass compared to that of the star, of mass M_* , the hydrostatic balance equation takes the form

$$\frac{dP}{dr} = -\frac{GM_*\rho}{r^2}. \quad (95)$$

Solving (95) to find the dependencies of the pressure and density on r , assuming these vanish at $r = R_*$, we obtain

$$\rho = \rho_* \left(\frac{R_*}{r} - 1 \right)^{\frac{1}{\Gamma_1 - 1}}, \text{ with } \rho_* = \left(\frac{(\Gamma_1 - 1)GM_*}{K_c \Gamma_1 R_*} \right)^{\frac{1}{\Gamma_1 - 1}}. \quad (96)$$

Setting $N^2 = 0$ in equation (87), using the polytropic equation of state and equation (96) we obtain

$$\frac{d^2\Phi_c}{dr^2} + \frac{1}{(\Gamma_1 - 1)r(1 - r/R_*)} \frac{d\Phi_c}{dr} - \frac{\Lambda}{r^2} \Phi_c = 0, \quad (97)$$

where $\Phi_c = r^2 \rho \xi$. We look for a solution to (97), which tends to zero when $r \rightarrow R_*$. This solution is expressed through the hypergeometric function, $F(\alpha, \beta, \gamma, z)$, as

$$\Phi_c = C_c r^{\alpha} x^{\frac{\Gamma_1}{\Gamma_1 - 1}} F(b_1, b_2, b_3, x), \quad (98)$$

where $x = 1 - r/R_*$,

$$\begin{aligned} a &= \frac{1}{2} \left(-\frac{2 - \Gamma_1}{\Gamma_1 - 1} + \sqrt{\left(\frac{2 - \Gamma_1}{\Gamma_1 - 1} \right)^2 + 4\Lambda} \right), \\ b_{1,2} &= \frac{1}{2} (2a - 1 \pm \sqrt{1 + 4\Lambda}) + \frac{\Gamma_1}{\Gamma_1 - 1}, \text{ and} \\ b_3 &= \frac{2\Gamma_1 - 1}{\Gamma_1 - 1} \end{aligned} \quad (99)$$

and C_c is an arbitrary constant. As we discuss in the next section, for our purposes it is important to know the quantity

$$B_c = - \left(\frac{d\xi/dr}{\xi} \right)_{r=r_c}, \quad (100)$$

calculated for the solution discussed above, at the base of the convective envelope, as a function of Λ . This quantity can be calculated from (98). It will be seen from the above analysis that this quantity depends on the eigenfrequency only through its dependence on Λ . The dependence B_c on Λ , for $\Gamma_1 = 5/3$ and $r_c/R_* = 0.7$, is illustrated in Fig. 5.

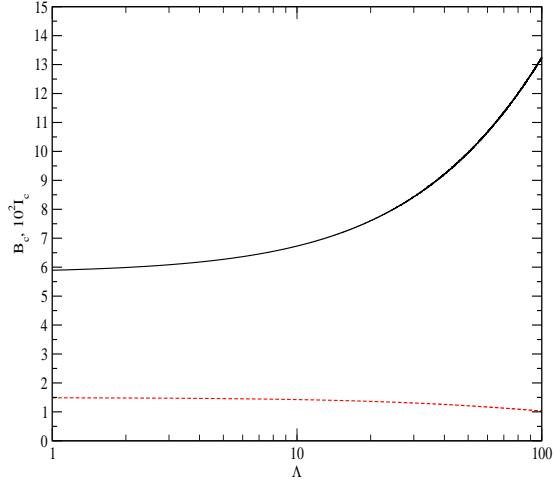


Figure 5. The quantities B_c (100) and $10^2 I_c$ (see (118)) determined from the solution (98) as functions of Λ for $r_c = 0.7R_*$. B_c is shown as the solid curve. Note that the values of B_c and I_c found from the asymptotic analysis are in quite good agreement with those obtained using numerically calculated eigenfunctions.

6.1.4 Matching conditions and eigenfrequencies

In order to match the solutions in the radiative zone and the convective envelope we use the solution (91) in the limit $z \rightarrow 0$ to calculate $-(d\xi/dr)/\xi$ as the base of the convective envelope is approached from below. Then, we equate the resulting expression to (100) which gives the same quantity as the base of the convective envelope is approached from above. This gives a compatibility relation, which will be satisfied only when ω is an allowed eigenfrequency. In the limit $z \rightarrow 0$ equation (91) gives

$$\begin{aligned} \xi &\approx C_1 (K/(2\omega))^{1/(2+q)} \frac{x}{\Gamma((3+q)/(2+q))} + \\ &C_2 (K/(2\omega))^{-1/(2+q)} \frac{1}{\Gamma((1+q)/(2+q))}, \end{aligned} \quad (101)$$

where $\Gamma(z)$ denotes the gamma function. We then find $-(d\xi/dr)/\xi$ in the limit $z \rightarrow 0$ from (101) and equate it to the same quantity determined from the solution in the convective envelope given by equation (100).

Equation (92) can be used to specify the ratio C_1/C_2 which, after its elimination from the expression, yields a compatibility condition from which the eigenfrequencies can be determined. This compatibility condition can be written down as

$$\frac{\cos(\phi_c - \pi q/(8+4q))}{\cos(\phi_c - \pi(4+q)/(8+4q))} = -\frac{\Gamma((3+q)/(2+q))}{\Gamma((1+q)/(2+q))} B_c \bar{\omega}^{\frac{2}{2+q}}, \quad (102)$$

where $\bar{\omega} = 2\omega/K$, and ϕ_c is given by equation (93).

In the limit $\omega \rightarrow 0$ the term on the right hand side of (102) is small. This allows us to solve equation (102) iteratively. First we set the term on the right hand side to zero.⁷ Then we find that we

⁷ This is equivalent to setting $C_1 = 0$ which corresponds to the radial

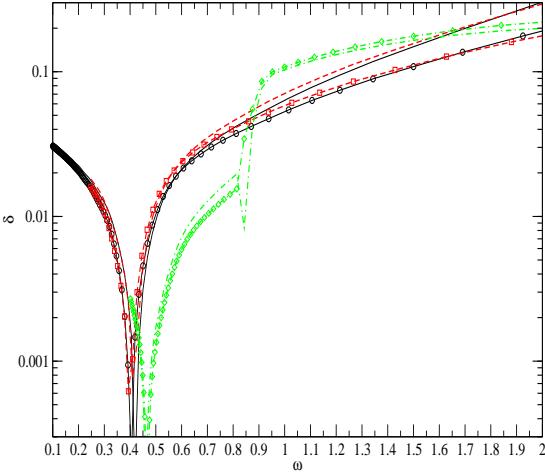


Figure 6. The difference $\delta = |\omega_{WKBJ} - \omega_n|/\omega_n$ between the WKBJ eigenfrequencies ω_{WKBJ} and the corresponding frequencies determined numerically ω_n . The results are plotted as a function of ω_n calculated for the CD model of the present day Sun and expressed in units of $\sqrt{GM_*/R_*^3}$. The value of I defined in equation (104) is approximately equal to $14.91\sqrt{\frac{CM_*}{R_*^3}}$. Since all the fitting models shown in table 1 give quite similar curves we show only the results corresponding to the CD2 model. See the text for the description of different curves.

must have $\phi_c = \pi(n + 1/2 + q/(4(2+q)))$, where n is an integer. Using this result in equations (89) and (93) we obtain

$$\omega \approx \omega_0 = \frac{\sqrt{\Lambda}I}{\pi(n + (1+l)/2 + q/(4(2+q)))}, \quad (103)$$

where

$$I = \int_0^{r_c} \frac{dr}{r} N. \quad (104)$$

Now let us take account of a small non zero B_c by setting $\phi_c = \pi(n + 1/2 + q/4(2+q)) + \Delta\phi$, where $\Delta\phi$ is assumed to be small. We substitute this expression into (102) and solve the resulting equation for $\Delta\phi$ using perturbation theory to linear order, thus obtaining

$$\Delta\phi = B_c(2\omega/K)^{2/(2+q)} \frac{\Gamma((3+q)/(2+q))}{\Gamma((1+q)/(2+q))} \sin(\pi/(2+q)), \quad (105)$$

and accordingly

$$\omega = \frac{\sqrt{\Lambda}I}{\pi(n + (1+l)/2 + q/(4(2+q))) + \Delta\phi(\omega_0)}. \quad (106)$$

We recall that for rotating stars equation (106) gives only an implicit equation for eigenfrequencies since in that case Λ is a function of ω .

We remark that for low frequencies the influence of the convective envelope solution on the spectrum, through B_c , provides

eigenfunction having its derivative close to zero at the base of convection zone

only a small correction to the eigenspectrum. This is consistent with the idea that the properties of the rotationally modified g mode spectrum are almost completely determined by the radiative zone.

We now compare results obtained from our analytic expression for the eigenfrequencies given by equation (106) with those obtained from an expression for ω given in Berthomieu et al (1978) and also with those obtained from numerical calculations for the CD model of the Sun in Fig. 6. The simple expression obtained by Berthomieu et al (1978) differs from equation (106) by replacing the denominator in that equation with $\pi(n + \frac{1}{2}(1 + \sqrt{\Lambda + \frac{1}{4}}))$.

In Fig. 6 we show the quantity $\delta = |\omega_{WKBJ} - \omega_n|/\omega_n$, where ω_{WKBJ} and ω_n stand for the eigenfrequencies determined from the WKBJ analysis and numerical calculation respectively, as a function of ω_n expressed in units of $\sqrt{GM_*/R_*^3}$. The curves with symbols showing positions of the eigenfrequencies are calculated from our analysis, while the curves with no symbols are for eigenfrequencies determined from the expression of Berthomieu et al (1978). The solid, dashed and dot dashed curves are for $\Omega = 0$, $\Omega = 0.42\Omega_*$ and $\Omega = -0.42\Omega_*$, respectively. It is seen that the deviation between the numerical and analytical results does not vary monotonically with δ increasing as ω becomes very small. This feature is most probably a consequence of numerical errors. The curves corresponding to $\Omega = 0$ and $\Omega = 0.42\Omega_*$ are similar to each other, this can be explained by the rather weak dependence of Λ on ν when $\nu = \frac{2\Omega}{\omega}$ is positive, see Fig. 1. Since the WKBJ approximation gets better with increasing numbers of nodes n , which is, in its turns, gets larger with increasing Λ , the deviation δ corresponding to those curves is appreciably larger than the deviation corresponding to retrograde rotation at $\omega < 0.8$ because Λ increases significantly with decreasing ω in that case, see Fig. 1.

As seen from Fig. 6 our approach gives deviations from the numerical results that are smaller than those of Berthomieu et al (1978) for prograde rotation and no rotation. However, the difference is typically rather moderate being of the order of several tens per cent. For the case of retrograde rotation and large values of ω the Berthomieu et al. (1978) expression gives somewhat better agreement though the difference is small. When $\omega < 0.847\Omega_*$ our approach is closer to the numerical one than that of Berthomieu et al. (1978). Note a non-monotonic behaviour of the curve corresponding to the Berthomieu et al (1978) expression close to $\omega = 0.847\Omega_*$. This value of ω corresponds to $\nu = 1$, which is a special value of this parameter delimiting the modes dominated by the gravity ($\nu < 1$) and Coriolis forces ($\nu > 1$). We suspect that this feature may be purely numerical because of the behaviour of the singularities of equation (62), determining Λ , as ν passes through unity.

For our estimates of the overlap integrals given below it is convenient to substitute the explicit expression for the phase ϕ_c calculated above into equations (92) for the constants C_1 and C_2 . Making use of (105) to do this, we easily obtain

$$\begin{aligned} C_1 &= (-1)^n \sqrt{\frac{\pi K}{2\omega}} B_c \left(\frac{2\omega}{K} \right)^{2/(2+q)} \frac{\Gamma((3+q)/(2+q))}{\Gamma((1+q)/(2+q))} C_*, \\ C_2 &= (-1)^n \sqrt{\frac{\pi K}{2\omega}} \left(1 - B_c \left(\frac{2\omega}{K} \right)^{2/(2+q)} \times \right. \\ &\quad \left. \frac{\Gamma((3+q)/(2+q))}{\Gamma((1+q)/(2+q))} \cos(\pi/(2+q)) \right) C_*. \end{aligned} \quad (107)$$

6.2 Asymptotic expressions for the normalised overlap integrals \hat{Q}_j

Having determined eigenfrequencies and eigenfunctions, we can use them to estimate the normalised overlap integrals determining the strength of tidal interactions. We evaluate the unnormalised Q derived from (47), and the norm n given by (85) (from now on we omit the mode subscripts (i, j) below unless absolutely necessary).

The overlap integral is written as $Q = \alpha Q_r$. The factor α , coming from the angular integration is calculated numerically and, as it does not depend on the radial form of the eigenfunctions, it is assumed to be known function of frequency (see Fig. 1). As we are working in the low frequency limit, we use the reduced form (80) to evaluate the factor Q_r . It is important to note that the evaluation of this expression can be simplified for solar type stars. For these, the main contribution to the integral comes from, either the transition region between the radiative core and the convective envelope, or from the convective envelope itself. We discuss these contributions separately below. Note that in view of various integrations by parts, these contributions do not necessarily relate to physical contributions from the regions concerned.

In what follows we adopt natural units, expressing radii in units of the stellar radius R_* , mass in units of M_* , frequencies in units of $\sqrt{GM_*/R_*^3}$ and density in units of the stellar mean density $\rho_* = 3/(4\pi)(M_*/R_*^3)$. In these natural units the factor GM_* in (111) can be set equal to 1. The normalised overlap integrals have dimension of $\sqrt{M_*R_*}$. Accordingly we express them in these units. From equation (47), given the fact that we express the density in units of the mean stellar density, it follows that expressions for the normalised overlap integrals obtained with quantities expressed in our units, must be additionally multiplied by factor of $\sqrt{3/(4\pi)}$. This will be assumed implicitly.

6.3 An expression for the norm

We consider the expression (85) which gives the norm n written as the sum of two contributions such that $n = n_{st} + n_r$. In the limit $\omega \rightarrow 0$, the main contribution to the radial integrals in (85) is expected to come from the region in the radiative core where the radial wavelength and group velocity are smallest and accordingly radial displacement is given by the WKBJ expression (88). In the same limit and in the same region of the star, $|\xi^S| \gg |\xi|$ as expected for low frequency modes in a stratified medium. Thus it follows from equations (64), (65) and (88) that

$$\begin{aligned} \xi^S &\approx \frac{r}{\Lambda} \xi' \approx \frac{C_{WKBJ}}{\Lambda^{1/2}\omega} \rho^{-1/2} r^{-3/2} N^{1/2} \cos(\Psi_{WKBJ}) \quad \text{with} \\ \Psi_{WKBJ} &= \frac{W}{\omega} + \phi_{WKBJ}. \end{aligned} \quad (108)$$

Substituting this expression into the integrand determining n_{st} in (85), neglecting the contribution proportional to ξ there and adopting the average value of $\cos^2(\Psi_{WKBJ}) = \frac{1}{2}$ we get

$$n_{st} \approx \frac{C_{WKBJ}^2}{2\omega^2} I, \quad (109)$$

where I is given by equation (104). Since $n_r = \nu I_r I_\theta$, with the integral I_r , defined through (84), readily being seen to be approximately equal to n_{st}/Λ on account of $|\xi^S| \gg |\xi|$, we finally obtain

$$n = n_{st} + n_r \approx \frac{C_{WKBJ}^2}{2\omega^2} I \left(1 + \frac{\nu I_\theta}{\Lambda} \right), \quad (110)$$

where I_θ is defined through equation (84). We recall that in general the latter has to be calculated numerically.

6.4 The estimate of the overlap integral and Q_r

In order to calculate the integral in the expression for Q_r , we use the approximate expression (80). The integrand in that expression is proportional to the function $h(r)$ defined there. We remark that the main contribution to the integral comes from regions close to the boundary between the radiative zone and the convective envelope where the eigenmodes have their largest scale. The term proportional to the Brunt - Väisälä frequency in (80) is small in these regions and we set it to zero. Also, as indicated above, the mass of the star may be taken to be constant in these regions so that $g \approx GM_*/r^2$. Substituting this into the expression for $h(r)$ in (80), we obtain the useful and simplified form

$$h(r) \approx \frac{r^2}{GM_*} \left(1 - \frac{30}{\Lambda} \right). \quad (111)$$

The integral may be regarded as being composed of two contributions, which should be evaluated separately, the first comes from the radiative region close to the base of convective envelope, and the second from the envelope itself.

6.4.1 The contribution to the integral from the radiative region close to the base of the convective envelope

In order to calculate this contribution we set all quantities in the integrand, multiplying the radial displacement ξ , equal to their values at $r = r_c$ and assign the subscript (c) to them, e.g. $\rho_c = \rho(r_c)$. We assume that the distance from the base, $x = r_c - r > 0$, is small and change the integration variable from x to z as defined in equation (91). We substitute ξ as given by (91) into the integral in (80). Taking the lower limit to be zero and formally extending the upper limit of integration to infinity gives the contribution from the radiative region. Proceeding in this way we get a linear combination of two integrals containing Bessel functions and powers of z . These integrals can be evaluated by standard methods using Gradshteyn & Ryzhik (2007) giving the contribution to Q_r from the radiative zone, Q_{rr} as

$$\begin{aligned} Q_{r,r} &= \left(1 - \frac{30}{\Lambda} \right) \frac{2^{3/(2+q)}}{(2+q)} \rho_c r_c^6 D_c^{3/2} \omega^2 \times \\ &\quad \left(C_1 \frac{\Gamma(2/(2+q))}{\Gamma((1+q)/(2+q))} + C_2 \frac{\Gamma(1/(2+q))}{\Gamma(q/(2+q))} \right), \end{aligned} \quad (112)$$

where $D_c = (\omega/K)^{2/(2+q)}$ and we recall that K is defined in (93). By construction this expression does not include the contribution to the integral from the convective envelope which is considered separately below. Next we substitute C_1 and C_2 as given by (107) and use (92) to replace C_* by C_{WKBJ} in the expression for $Q_{r,r}$. In this way we obtain

$$\begin{aligned} Q_{r,r} &= \left(1 - \frac{30}{\Lambda} \right) Q_b \left(\frac{\Gamma(1/(2+q))}{\Gamma(q/(2+q))} (1 - \Delta_{corr} \cos(\pi/(2+q))) \right. \\ &\quad \left. + \Delta_{corr} \frac{\Gamma(2/(2+q))}{\Gamma((1+q)/(2+q))} \right), \end{aligned} \quad (113)$$

where

$$\begin{aligned} \Delta_{corr} &= \frac{\Delta\phi}{\sin(\pi/(2+q))} \\ &= B_c \left(\frac{2\omega}{K} \right)^{2/(2+q)} \frac{\Gamma((3+q)/(2+q))}{\Gamma((1+q)/(2+q))}, \end{aligned} \quad (114)$$

see equation (107), and

$$Q_b = \frac{(-1)^n (K/2)^{-\frac{4-q}{2(2+q)}} \sqrt{\pi}}{(2+q) A^{1/4}} \rho_c^{1/2} r_c^{9/2} \omega^{\frac{3(4+q)}{2(2+q)}} C_{WKBJ}. \quad (115)$$

Expressions (113-115) determine the contribution of the radiative region close to the convective base to the overlap integral. Note that $\Delta_{corr} \ll 1$ when ω is small. It thus represents a correction to the leading term.

6.4.2 Contribution to the overlap integral from the convective envelope

In order to find this contribution we first determine the constant C_c that scales the solution (98), applicable in the convective envelope, in terms of the constant C_2 that scales the solution in the radiative region. This can be done by applying the condition that the radial displacement ξ be continuous at $r = r_c$. Using (98) and (101) evaluated at $x = 0$ to obtain the displacements just outside and just inside $r = r_c$ respectively, we find that the continuity condition gives

$$C_c = C_2 (K/2\omega)^{-\frac{1}{2+q}} \frac{1}{\Gamma((1+q)/(2+q))\zeta}, \quad (116)$$

where $\zeta = (r_c^{a-2}/\rho_c)x_c^{\frac{1}{\Gamma_1-1}} F(b_1, b_2, b_3, x_c)$ and $x_c = 1 - r_c$. After using (116) to substitute for C_c in (98) we substitute the result into (80). Making use of (92), (107) and (111) we evaluate the integral to obtain

$$Q_{r,c} = (1 - \frac{30}{\Lambda})(1 - \Delta_{corr} \cos(\frac{\pi}{2+q}))Q_{b,c}, \quad \text{where} \\ Q_{b,c} = (-1)^n \frac{\sqrt{\pi\rho_c} (K/2)^{\frac{q}{2(2+q)}}}{\Gamma((1+q)/(2+q))r_c^{3/2} A^{1/4}} I_c \omega^{\frac{8+3q}{2(2+q)}} C_{WKBJ}, \quad (117)$$

with

$$I_c = \frac{\int_{r_c}^1 dr r^{4+a} x_c^{\frac{1}{\Gamma_1-1}} F(b_1, b_2, b_3, x)}{r_c^{(a-2)} x_c^{\frac{1}{\Gamma_1-1}} F(b_1, b_2, b_3, x_c)}. \quad (118)$$

The integral $I_c(\Lambda)$ is plotted as a function of Λ in Fig. 5. It will be seen that I_c depends rather weakly on Λ in the considered range, being approximately equal to its value for a non-rotating star, namely $1.45 \cdot 10^{-2}$.

Note that when Λ is assumed to be constant (as in the non-rotating case) and $q = 1$, from (117) it follows that $Q_r \propto \omega^{\frac{11}{6}}$. This scaling was obtained by Zahn (1970) using a different formalism.

6.4.3 Analytic expressions for the normalised overlap integrals

The normalised overlap integrals entering the expressions for the tidal energy and angular momentum transfer given by (50) are related to Q_r , α , and n , through

$$\hat{Q}_j = \alpha \hat{Q}_r, \quad \text{where} \quad \hat{Q}_r = \frac{Q_r}{\sqrt{n}}. \quad (119)$$

Note that α is defined through equation (70) and its form for the dominant mode ($i = 1$) is shown in Fig. 1. The quantities Q_r can be specified as $Q_r = Q_{r,r} + Q_{r,c}$, where $Q_{r,r}$ and $Q_{r,c}$ are given in equations (113) and (117), respectively and we recall that n is given by (110).

In order to evaluate these expressions, along with α , we require Λ , both of which are shown in Fig. 1, B_c , and I_c , the latter two quantities being plotted in Fig. 5. In addition to these we

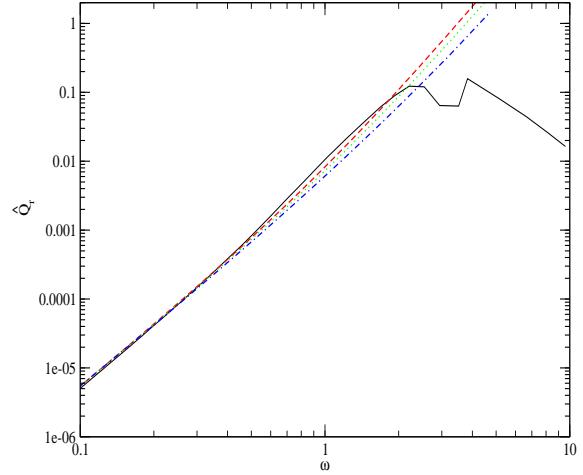


Figure 7. The dependence of \hat{Q}_r on ω for the non-rotating CD model with different fitting models for the radial dependence of the Brunt - Väisälä presented in table 1. See the text for a description of the different curves.

require the stellar model dependent quantities, I , A , q , r_c , and ρ_c . These are specified below and in table 1. The above information enables straightforward calculation of analytic estimates for the normalised overlap integrals for rotating stars, which match well the numerically determined ones, for forcing frequencies less than the critical rotation period.

The values of the integral I are taken to be equal to 14.91 and 5.898 for the CD and IR models, respectively. The density ρ_c , in standard units, at the base of convection zone is equal to 0.1048 in the CD model and to 0.1375 in the IR model. It is assumed that the base of the convection zone is situated at $r_c = 0.729$ in all cases.

6.5 Quantitative comparison of the the overlap integrals obtained numerically and analytically

The comparison of the analytic estimates for the overlap integrals determined from equation (119) with values obtained numerically is illustrated in Figures 7-11.

In Figs 7 and 8 we show \hat{Q}_r as a function of ω for the non-rotating CD and IR models, respectively. For non-rotating models we straightforwardly have $\hat{Q} = \hat{Q}_r$. The solid curves give numerically evaluated overlap integrals with the uppermost one corresponding to the IR model. For the non-rotating models of the star the Cowling approximation is relaxed. The dashed, dotted and dot dashed curves give our analytically determined overlap integrals for the CD2 and IR2 fitting models (dashed curves), for the CD1 and IR1 fitting models (dotted curves) and for the CDL and IR1L models (dot dashed curves), respectively. As seen from this Fig. when ω is small our analytic results are an excellent agreement with the numerically determined ones, especially for the CD2 and IR2 fitting models, which will be used later in other Figures. For the CD2 model the disagreement is smaller than or of the order of 10 per cent when $\omega \leq 0.5$ and smaller than 30 per cent when $\omega \leq 1$. For

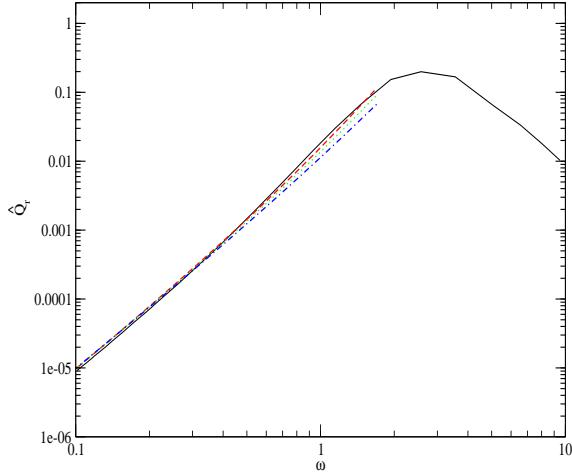


Figure 8. As in Fig. 7 but for the IR model.

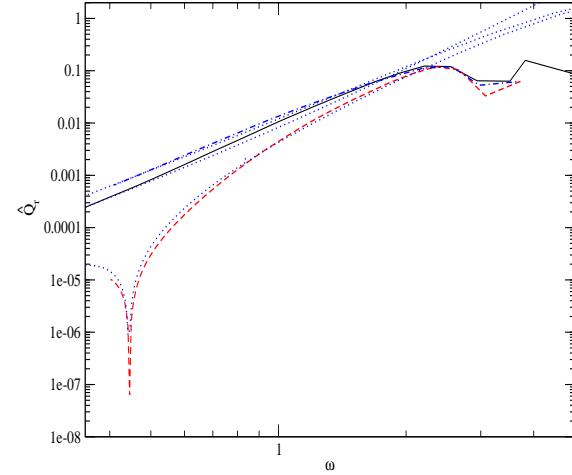


Figure 10. As in Fig. 7 but showing \hat{Q}_r for a rotating star with $\Omega_r = \pm 0.42$. All curves are for the CD model with the CD2 fitting model for the radial dependence of the Brunt - Väisälä frequency.

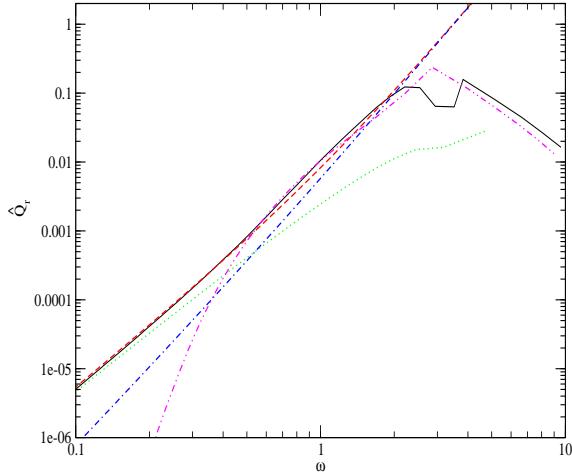


Figure 9. As in Fig. 7, but showing the contribution of different terms to \hat{Q}_r . All curves are for the CD model with the CD2 fitting model for the radial dependence of the Brunt - Väisälä frequency.

the IR model the disagreement is even smaller being of the order of or smaller than 10 per cent when $\omega \leq 1.5$.

It is very important to stress that the IR model, describing a young zero age main sequence star of one solar mass, has larger overlap integrals. This means that tidal interactions are stronger in that case. Our analytic approach to the calculation of overlap integrals allows us to explain this as follows. One can check that the main difference in the values of the overlap integrals is determined

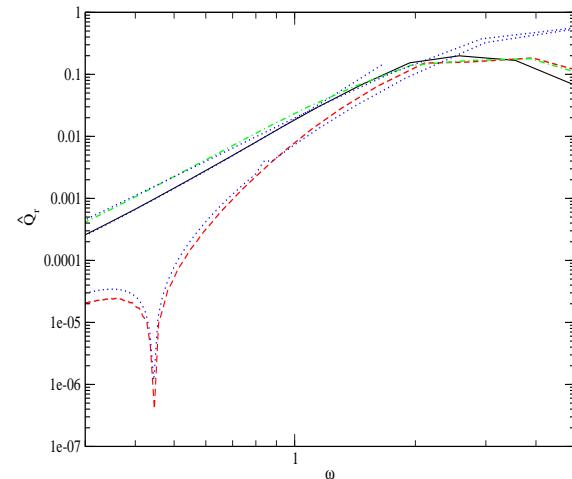


Figure 11. As in Fig. 10 but for the IR model of the star with the IR2 fitting model for the radial dependence Brunt - Väisälä frequency.

by the the integral I entering in the expression for the norm (110). As mentioned above, we have $I = 14.91$ and 5.83 for the CD and IR models, respectively. Our analytically determined overlap integrals are inversely proportional to square root of I . Therefore, the overlap integrals of the IR models are larger than the ones of the CD models by a factor approximately given by $\sqrt{14.91/5.83} \approx 1.6$. Since the energy and angular momentum exchanges through tidal

interactions are proportional to square of the overlap integrals, in the case of the zero age MS star, these are approximately 2.55 times larger in comparison with those obtained for the present day solar model.

In Fig 9 we show different contributions to \hat{Q}_r , for the CD2 model, separately. The solid and dashed curves are as in Fig. 7. The dot dashed curve shows the analytically determined \hat{Q}_r , but with $Q_{r,c}$ set to zero in (119). The dotted curve shows \hat{Q}_r , but with $Q_{r,r} = 0$ in (119). Thus, the dot dashed and dotted curves show the separate contributions of the radiative region close to the base of convection zone and the contribution from convection zone i to \hat{Q}_r . As seen from Fig. 9, $Q_{r,r}$ is more important for relatively large values of ω , while the $Q_{r,c}$ gives a contribution that becomes increasingly important in the limit of $\omega \rightarrow 0$. However, for the range of ω shown, both contributions must be taken into account in order to obtain good agreement between the numerical and analytical results. Additionally, we show the overlap integrals of a star modelled as an $n = 3$ polytrope, plotted using the dot dot dashed curve. Unlike the overlap integrals for the the realistic star, the polytropic ones decrease exponentially with decreasing ω . Therefore, when g modes of sufficiently high order determine the tidal exchange of energy and angular momentum, this is much smaller for a polytropic star.

In Fig. 10 we compare numerical and analytic results for a rotating CD2 model. In order to delineate the effects of rotation, the rotational frequency is taken to be large with $|\Omega_r| = 0.42$. The solid, dashed, and dot dashed curve show the numerical results corresponding to $\Omega_r = 0, -0.42$ and 0.42 , respectively. The dotted curves correspond to the analytic calculations and are closest to their corresponding numerical curves. One can see that the agreement between analytical and numerical results is again very good for $\omega < 1$ for the non-rotating star and the star with prograde rotation. The analytic and numerical curves can hardly be distinguished. For the case of retrograde rotation the curves disagree by up to 30 per cent in the range shown. The curves corresponding to prograde rotation are qualitatively similar to the curves corresponding to a non-rotating star, the only difference being that they indicate somewhat larger values of \hat{Q}_r for a given ω . The curve corresponding to retrograde rotation behaves in a non-monotonic manner. It has a strong minimum at $\omega = 0.44$, which corresponds to $\nu = -1.9$. From Fig. 1 it follows that $\Lambda = 30$ for this value of ν and from equation (111) it follows that the factor $h(r)$ entering our analytic expression for \hat{Q}_r is then equal to zero. This explains the non-monotonic character of the behaviour of \hat{Q}_r for retrograde rotation. In general, in this case the radial overlap integral is smaller than for the non-rotating star. This as well as the decrease of the 'angular overlap integral' α for retrograde rotation (see Fig. 1) suppresses the tidal transfer of energy and angular momentum for sufficiently large retrograde rotation. Note, however, that for retrograde rotation there are other factors, such as a match between forcing and mode frequencies that tends to occur for lower order modes in comparison to the prograde case, that may amplify the transfer, see e.g. Ivanov & Papaloizou (2011) . In Fig. 11 the corresponding results are given for the rotating IR2 model. The results are qualitatively similar to the previous case.

In Figs 12 and 13 we show the dependence of the overlap integral defined through (119) (the index j being suppressed) with ω set equal to 1 in (113) and (117) on ν , for the CD2 and IR2 models, respectively. Note that the overlap integral given by (119) includes the factor contributed by the angular overlap integral $\alpha(\nu)$. The solid curves show \hat{Q} , the dashed curves show \hat{Q} calculated with the contribution from the convective envelope, $Q_{r,c}$ defined in

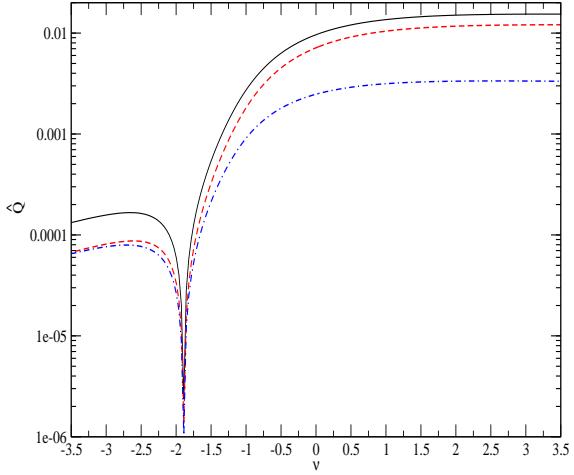


Figure 12. The value of the overlap integral (119) for a fixed eigenfrequency $\omega = 1$ plotted as a function of ν together with the contributions to it from the regions just inside the radiative core and the convective envelope. The CD2 fitting model is used. See the text for a description of the different curves.

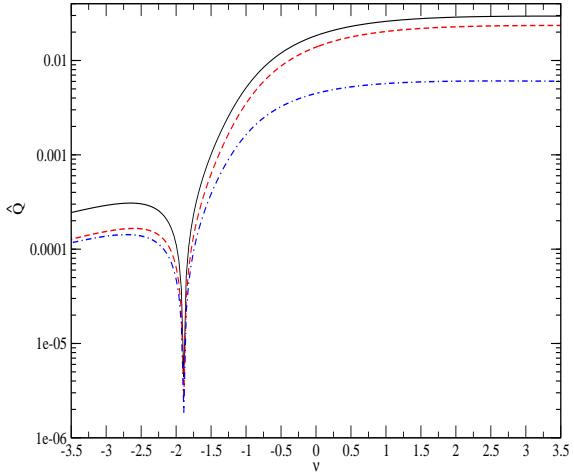


Figure 13. As for Fig. 12 but for the IR2 model.

(117) set to zero and the dot-dashed curves show \hat{Q} calculated with the contribution from the radiative interior, $Q_{r,r}$ defined in (113) set to zero. One can see from Figs 12 and 13 that at negative values of ν , \hat{Q} becomes quite small. The minimum at $\nu \approx -1.9$ is determined by the fact that $\Lambda \approx 30$ there and the integral is strongly suppressed as discussed above.

As follows from equations (113) and (117) the contributions to

the overlap integral coming from the radiative interior near base of convection zone and the convective envelope are, to a good approximation, proportional to powers of ω . This can be used to extend the results presented in Figs. 12 and 13 to other values of ω by making appropriate scaling. Of course, when using such results, one has to take into account that the values of ω , the mode order and Ω are linked together through equation (106) and also the definition of ν .

7 TIDAL EVOLUTION TIMESCALES FOR A BINARY WITH A SMALL ECCENTRICITY AND WITH ORBITAL AND SPIN ANGULAR MOMENTA ALIGNED IN THE MODERATELY LARGE DISSIPATION REGIME

As a simple illustration of application of our formalism we calculate the timescales for the evolution of the orbital elements of a binary system induced by tidal interaction. We consider a binary consisting of a primary star of mass M_* on which tides are exerted by the secondary of mass M_2 that is approximated as being a point mass. For simplicity, we assume below that the orbit has a small eccentricity, e , and the stellar rotation frequency is small in comparison with the characteristic frequency Ω_* .

We shall assume that the action of dissipation on the eigenmodes is sufficiently strong for the system to be in the regime of moderately strong viscosity discussed above. Then we can use (51) to calculate the orbital energy dissipation rate due to tides. In the case of Sun-like stars considered here, this may require the excited waves to be of large enough amplitude to break as they propagate towards the stellar centre (see Barker 2011 and references therein for a discussion). Although such a condition may not be satisfied, assuming that it is should provide the most optimistic estimate of the potential long term tidal evolution due to the excitation of these modes. Note too that the action of turbulence in convection zones is likely to be ineffective for the short orbital frequencies we consider on account of its long characteristic time scale.

This problem has been discussed in a similar setting by e.g. Zahn (1977) and Goodman & Dickson (1998). However, they did not include the effects of Coriolis forces, taking account of rotation only through a shift in tidal forcing frequency. Our formalism includes the effects of Coriolis forces under the traditional approximation and so allows us to assess their significance. Moreover, the formalism of Zahn (1977) has been mainly applied to more massive stars having a convective core and radiative envelope in contrast to the solar type stars explicitly considered here (see also Papaloizou & Savonije 1997). Although our general approach is applicable to this case, the asymptotic calculation of the overlap integrals needs to be considered separately.

In sections 2 - 4, the stellar angular velocity was taken to be positive so that prograde normal modes had positive frequencies while retrograde normal modes had negative frequencies. However, for the discussion of normal modes and overlap integrals from section 4 onwards, it was convenient to exploit the symmetry that enabled retrograde modes to have a positive frequency, with the sign of Ω being reversed to become negative. In this section we consider the tidal interactions of a star with a definite positive angular velocity, accordingly from now on we revert to the convention of section 2 - 4 for which retrograde modes have negative frequencies.

7.1 Equations for the evolution of the semi-major axis and eccentricity

Equations governing the evolution of the semi-major axis a , and the eccentricity, e , follow from consideration of the conservation of energy and angular momentum of the system. We begin by noting that tidal interaction with M_* induces rates of change of energy, \dot{E}_I , and angular momentum, \dot{L}_c , for the orbit given by equation (51). Conservation of energy and angular momentum implies that these produce changes in a and e according to

$$\frac{G\mu M_*^2 \dot{a}}{2a^2} = \dot{E}_I, \quad \frac{G\mu M_*^2 e \dot{e}}{a} = \dot{E}_I(1 - \frac{e^2}{2}) - \Omega_{orb} \dot{L}_c. \quad (120)$$

Here the eccentricity is assumed to be small so that powers of e higher than second order may be neglected and $\mu = M_2/M_*$ is the mass ratio. In what follows, we find it convenient to rewrite these equations in the form

$$\frac{\dot{a}}{a} = -\frac{2}{T_a}, \quad \frac{\dot{e}}{e} = -\frac{1}{T_e}, \quad (121)$$

where

$$T_e = -\frac{G\mu M_*^2 e^2}{a \dot{E}_e}, \quad T_a = -\frac{G\mu M_*^2}{a \dot{E}_I} \quad \text{and} \quad (122)$$

$$\dot{E}_e = \dot{E}_I(1 - \frac{e^2}{2}) - \Omega_{orb} \dot{L}_c. \quad (123)$$

When the rotation frequency Ω is specified we can use equations (51) and the expressions for the coefficients $A_{m,k}$ given in Appendix A to estimate the time scales T_a and T_e . We consider below two important cases 1) the case of synchronous rotation $\Omega = \Omega_{orb}$ and 2) the case of a non-rotating primary $\Omega = 0$, labelling the corresponding time scales by indices (1) and (2), respectively. Since we shall use dimensionless units, we write $\tilde{Q} = \sqrt{M_*} R_* \tilde{Q}$, where \tilde{Q} is the dimensionless overlap integral, we recall that we have suppressed the index j .

7.2 The case of synchronous rotation

Here we have $\Omega = \Omega_{orb}$. As indicated in Appendix A it turns out that to leading order in e , terms of higher order in e than linear in the expansion of the tidal potential may be neglected when substituting into the series (6). Thus, the term proportional $e^2/2$ in (123) results in a higher order contribution and can be neglected. We thus have $\dot{E}_e \approx \dot{E}_c$. The (m, k) pairs that remain to be considered are $(0, 1)$, $(2, 1)$ and $(2, 3)$. The conditions for resonance associated with a normal mode (41) associated with these are all included in the expression $\omega = \omega_{m,k} = k\Omega_{orb} - m\Omega = \pm\Omega_{orb}$. The corresponding values of the parameter $\nu = \pm 2$. As seen from Figs. 1 and 2 when either $\nu = -2$ and $m = 2$ or $\nu = \pm 2$ and $m = 0$ we have $\Lambda \approx 34$. This leads to a strong suppression of the corresponding overlap integrals determined by the factor $h(r) \propto 1 - \frac{30}{\Lambda}$, see the discussion above, equation (111) and the discussion of Fig. 10 for the $m = 2$ case. Therefore, their respective contributions to the energy and angular momentum exchange rates can be neglected and we take into account below only the dominant $(2, 3)$ term.

Equations (51) and (123) then give to the leading order in eccentricity

$$\begin{aligned} \dot{E}_e &= -\frac{147\pi^2 e^2}{40} \left(\frac{GM_2}{a^3} \right)^2 M_* R_*^2 \left[\frac{\tilde{Q}^2}{|d\omega/dj|} \right]_1, \quad \text{and} \\ \dot{E}_I &= 3\dot{E}_e, \end{aligned} \quad (124)$$

where functions of eigenfrequencies ω enclosed in square brackets

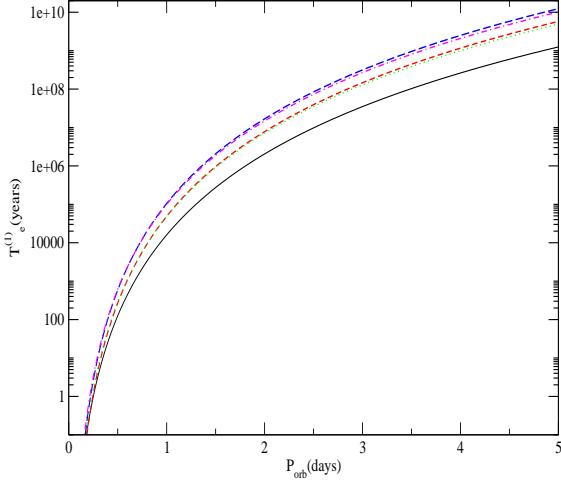


Figure 14. The evolution time scale $T_e^{(1)}$ calculated for the CD and IR models compared to the result of Goodman & Dickson (1998), for $\mu = 1$. See the text for the description of different curves.

[...] are assumed to be evaluated for $\omega = k\Omega_{orb}$ and $m = 2$ from now on.

The term $[(d\omega/dj)]_k$ can be evaluated setting $n = j$ in equation (106) and adopting the limit $n \gg 1$, noting that Λ varies only weakly with ν in the region of interest being close to 4.5 (see Fig. 1), with the result that

$$[(d\omega/dj)]_k = \frac{\pi k^2}{\sqrt{\Lambda I}} \frac{\Omega_{orb}^2}{\Omega_*}, \quad (125)$$

where the integral I is expressed in natural units. Substituting (125) in (124) we obtain

$$\dot{E}_I = -\frac{441\pi e^2 \dot{E}_*}{40} [\tilde{Q}^2]_1, \text{ and } \dot{E}_e = -\frac{147\pi e^2 \dot{E}_*}{40} [\tilde{Q}^2]_1,$$

where

$$\dot{E}_* = \sqrt{\Lambda I} \frac{G^2 \mu^2 \Omega_* M_*^3 R_*^2}{\Omega_{orb}^2 a^6}. \quad (126)$$

Now we use (126) to obtain the tidal evolution time scales given by (122) thus obtaining

$$\begin{aligned} T_e^{(1)} &= \frac{40T_*}{147\pi} [\tilde{Q}^{-2}]_1, \\ T_a^{(1)} &= \frac{1}{3} e^{-2} T_e^{(1)}, \end{aligned} \quad (127)$$

where

$$T_* = \frac{(1 + \mu)^{5/3}}{\sqrt{\Lambda} \mu I} \left(\frac{P_{orb} \Omega_*}{2\pi} \right)^{4/3} \Omega_*^{-1}, \quad (128)$$

and the orbital period $P_{orb} = 2\pi \Omega_{orb}^{-1}$. Thus, for this configuration we have $T_a \gg T_e$ when $e \ll 1$.

In order to calculate the overlap integral entering (127) for $m = 2$ we use the results of preceding section setting values of parameters determining the overlap integral to correspond to $\nu =$

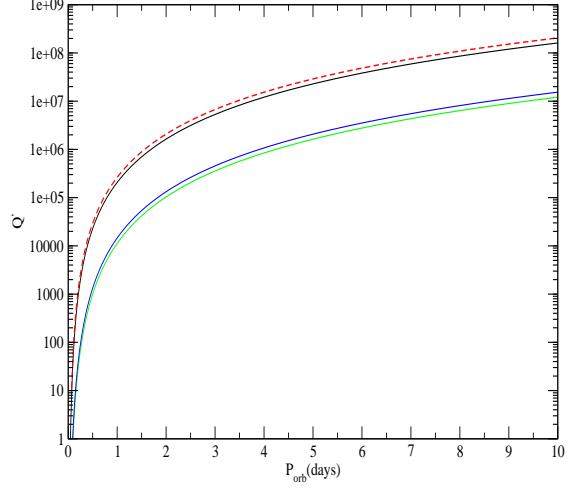


Figure 15. The tidal Q' parameter for orbital circularisation. The solid and dashed curves are respectively for CD2 and IR2 models in the case of synchronous rotation, the dotted and dot-dashed curves are respectively for the CD2 and IR2 models with no rotation.

2. Explicitly, we use $\alpha_1 \approx 1$, $\Lambda = 4.5$, $I_\theta = -0.1155$, $B_c = 6.2$ and $I_c = 1.4 \cdot 10^{-2}$.

We show the evolution timescales $T_e^{(1)}$ calculated for the CD and IR models in Fig. 14. The short dashed, dotted, long dashed and dot dashed curves are for the CD2, CDL, IR2 and IRL fitting models. One can see from Fig. 14 that there is little difference between the curves corresponding to different fitting models. The solid line plots a minor modification of the expression of Goodman & Dickson (1998) who give $T_e^{GD} = 8 \cdot 10^3 P_{orb}^7$, with P_{orb} in days, for a system of two tidally interacting Sun-like stars of equal mass.

Since we assume that the tides are raised only on one component, we plot $T_e^{GD} = 1.6 \cdot 10^4 P_{orb}^7 / \mu$ in Fig. 14 and others below. Our results are seen to give larger values of T_e for a given P_{orb} , e.g. for $P_{orb} = 4$ days, the results differ by a factor of five to ten. Thus we have $T_e \approx 2.7 \cdot 10^8$ yr, $1.15 \cdot 10^9$ yr and $2.35 \cdot 10^9$ yr from the modified Goodman & Dickson (1998) expression and for our calculations corresponding to the CD and IR models, respectively. We remark that Ogilvie & Lin (2004) indicate a possible reason why the Goodman & Dickson expression gives such an underestimate for T_e and we recall that the effects of Coriolis forces are not included. Note that the IR model gives larger values of T_e irrespective of the fact that its overlap integrals are larger than those of the CD model, for a given forcing frequency. This is due to a smaller radius of the early Sun compared to its present value. The younger star is more compact and, accordingly, less susceptible to tidal influence from a perturber.

Finally let us recall that for large values of P_{orb} , the appropriate normalised overlap integrals corresponding to the CDL and IRL fitting models are proportional to $P_{orb}^{-17/6}$. From equation (127) it follows that in this case $T_e \propto P_{orb}^7$ as for the Goodman & Dickson (1998) expression.

Finally we calculate the effective tidal Q' for the star by comparing our circularisation rate to that of Goldreich & Soter (1966).

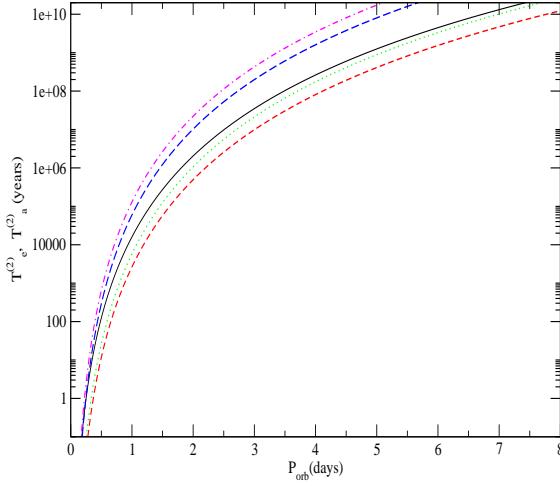


Figure 16. The evolution time scales $T_a^{(2)}$ and $T_e^{(2)}$ calculated for the CD and IR models compared the modified expression of Goodman & Dickson (1998). See the text for the description of the different curves.

This comparison leads to the expression

$$Q' = \frac{285}{98\pi\sqrt{\Lambda}I[\tilde{Q}^2]_1(1+\mu)^{1/2}} \left(\frac{2\pi}{P_{orb}\Omega_*} \right)^3. \quad (129)$$

We plot Q' as a function of orbital period for $\mu = 1$ in Fig. 15. This quantity is often assumed to be a constant. However, Fig. 15 shows that this varies between 10^6 and 10^8 as the orbital period varies from 1–10 days.

7.3 The case of non-rotating primary

Calculation of the time scales $T_a^{(2)}$ and $T_e^{(2)}$ proceeds analogously to the previous case. Since $\Omega = 0$, the forcing frequencies are of the form $\omega_{m,k} = k\Omega_{orb}$.

In order to calculate $T_a^{(2)}$ we can set $e = 0$ in the expressions for the potential expansion coefficients given in Appendix A. Then only the $A_{2,2}$ term with corresponding forcing frequency $\omega_{2,2} = 2\Omega_{orb}$ contributes to the series. The calculation is straightforward with the result that

$$\dot{E}_I = -\frac{3\pi\dot{E}_*}{40} [\tilde{Q}^2]_2, \quad (130)$$

and, accordingly,

$$T_a^{(2)} = \frac{40T_*}{3\pi} [\tilde{Q}^{-2}]_2 \quad (131)$$

It is convenient to separate the calculation of \dot{E}_e , and, accordingly, $T_e^{(2)}$ into two stages. First we evaluate $\dot{E}_I - \Omega_{orb}\dot{L}_c$. This quantity is determined by terms in the potential expansion with forcing frequencies $\omega_{0,1} = \Omega_{orb}$, $\omega_{2,1} = \Omega_{orb}$ and $\omega_{2,3} = 3\Omega_{orb}$. The respective coefficients $A_{m,k}$ are given in Appendix A. Substituting these into (51), noting that in the non rotating case, the overlap integrals are independent of m , we obtain

$$\dot{E}_I - \Omega_{orb}\dot{L}_c =$$

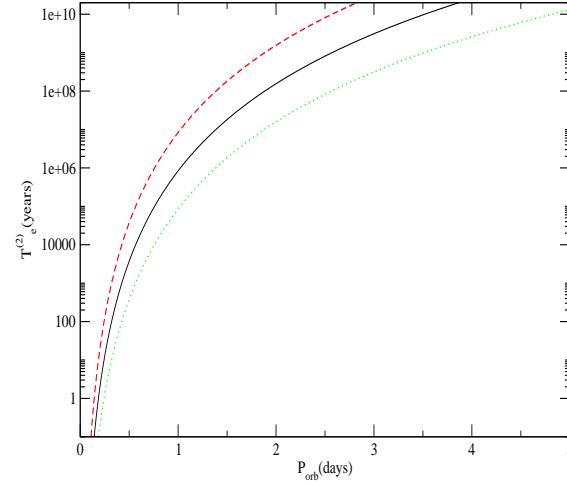


Figure 17. The evolution time scales $T_e^{(2)}$ plotted for $q = 10^{-3}$, $q = 10^{-4}$ and $q = 10^{-2}$ using the solid, dashed and dotted curves, respectively.

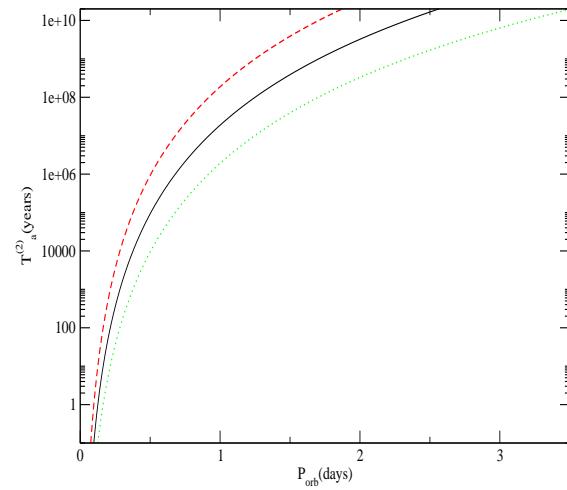


Figure 18. Same as Fig. 17 but $T_a^{(2)}$ is shown.

$$-\left(\frac{GM_2}{a^3}\right)^2 e^2 \left\{ \frac{3\pi^2}{20} \left[\frac{\hat{Q}^2}{|d\omega/dj|} \right]_1 + \frac{49\pi^2}{40} \left[\frac{\hat{Q}^2}{|d\omega/dj|} \right]_3 \right\} \quad (132)$$

Note that the term associated with the forcing frequency $3\Omega_{orb}$ gives the leading contribution since the normalised overlap integrals strongly decrease as $\omega \rightarrow 0$.

Now we calculate $e^2\dot{E}_I/2$. To leading order in e^2 it is sufficient to take into account only the term in the potential expansion

with forcing frequency $\omega_{2,2} = 2\Omega_{orb}$, thus from equation (130) we obtain

$$\frac{e^2}{2} \dot{E}_I = -\frac{3\pi^2}{8} \left(\frac{GM_2}{a^3} \right)^2 \left[\frac{\hat{Q}^2}{|d\omega/dj|} \right]_2 e^2. \quad (133)$$

From (123) and (132) we then find that

$$\dot{E}_e = -\frac{\pi^2}{20} e^2 M_* R_*^2 \left(\frac{GM_2}{a^3} \right)^2 F(\Omega_{orb}), \quad (134)$$

where

$$F(\Omega_{orb}) = \frac{49}{2} \left[\frac{\tilde{Q}^2}{|d\omega/dj|} \right]_3 - 3 \left[\frac{\tilde{Q}^2}{|d\omega/dj|} \right]_2 + \frac{15}{2} \left[\frac{\tilde{Q}^2}{|d\omega/dj|} \right]_1 \quad (135)$$

Now we make use of (125), applicable in this case, to eliminate the frequency derivatives with respect to j in (135) finally obtaining

$$\dot{E}_e = -\frac{\pi}{20} e^2 \dot{E}_* \mathcal{F}(\Omega_{orb}), \quad (136)$$

where

$$\mathcal{F}(\Omega_{orb}) = \left\{ \frac{49}{18} [\tilde{Q}^2]_3 - \frac{3}{4} [\tilde{Q}^2]_2 + \frac{15}{2} [\tilde{Q}^2]_1 \right\}, \quad (137)$$

and, accordingly,

$$T_e^{(2)} = \frac{20}{\pi} T_* / \mathcal{F}(\Omega_{orb}). \quad (138)$$

In Fig. 16 we show the characteristic times $T_a^{(2)}$ and $T_e^{(2)}$, for $\mu = 1$. The short dashed and dotted curves give $T_e^{(2)}$ for the CD2 and IR2 models respectively. The long dashed and dot dashed curves give $T_a^{(2)}$, respectively for the CD2 and IR2 models. The solid curves correspond to the modified Goodman & Dickson (1998) expression. We see that $T_a^{(2)}$ is much larger than $T_e^{(2)}$ in all cases. This is obviously due to the higher forcing frequency $3\Omega_{orb}$ effective in the latter case. It is interesting to note that the numerical values of $T_e^{(2)}$ are either smaller than, or close to the result of Goodman & Dickson (1998) regardless of the fact that our results were calculated for a non-rotating primary, while the Goodman & Dickson (1998) result corresponds to a primary in a state of synchronous rotation.

In Figs. 17 and 18 we show the evolution timescales T_e^2 and T_a^2 for mass ratios appropriate for exoplanetary systems. The solid, dashed and dotted curves correspond to $M_2 = M_J$, $0.1M_J$ and $10M_J$. As seen from these plots, dynamical tides in the regime of moderately large viscosity are potentially efficient only when orbital periods are rather small. Thus, the eccentricity is damped in a time less than 10^9 yrs only for $P_{orb} < 1.85, 2.56$ and 3.48 days, for $M_2 = 0.1M_J, M_J$ and $10M_J$, respectively. Similarly the semi-major axis can decay in a time less than 10^9 yrs only for $P_{orb} < 1.24, 1.69$ and 2.3 days respectively.

We obtain Q' for the star by comparing our circularisation rates \dot{e}/e to that of Goldreich & Soter (1966). In this case we find

$$Q' = \frac{285}{98\pi\sqrt{\Lambda}I[\tilde{Q}^2]_1(1+\mu)^{1/2}} \left(\frac{2\pi}{P_{orb}\Omega_*} \right)^3, \quad (139)$$

for the case of synchronous rotation and

$$Q' = \frac{855}{4\pi\mathcal{F}(\Omega_{orb})(1+\mu)^{1/2}} \left(\frac{2\pi}{P_{orb}\Omega_*} \right)^3 \quad (140)$$

in the non-rotating case. We plot Q' as a function of orbital period for $\mu = 1$ in Fig. 15. This quantity is found to vary between 10^4 and 10^7 as the orbital period varies from 1 – 10 days.

Finally we make a rough estimate of the effect of inertial waves that could be excited in the convective envelope in the case of

synchronous rotation. We note that Ogilvie & Lin (2007) calculate the Q' associated with inertial waves, for a solar model with a rotation period of ten days as a function of tidal forcing frequency as viewed in the rotating frame, appropriate to forcing with a $m = 2$ quadrupole potential (see their Fig.3). For the relevant forcing frequencies $\omega = \pm\Omega_{orb}$, $Q' \sim 10^8$, implying that the effect of inertial modes may be comparable to those due to rotationally modified gravity modes at the longest orbital periods considered ~ 10 days. But note that if the tidal dissipation due to these waves scales as Ω^2 as indicated in Ogilvie (2013), their effect may be somewhat less important at shorter orbital periods.

8 CONCLUSIONS AND DISCUSSION

In this paper we have calculated the general tidal response of a uniformly rotating star and used it to find the rate of transfer of energy and angular momentum between the star and the orbit. The results are expressed as a sum of contributions from normal modes each being proportional to the square of the associated overlap integral.

We considered the response due to an identifiable regular dense spectrum of normal modes such as one provided by the low frequency rotationally modified gravity modes. We obtained expressions (51) for the energy and angular momentum transfer rates appropriate to the moderately viscous case for which the waves associated with the modes are damped before reaching an appropriate boundary (the centre for a radiative core and the surface for a radiative envelope) or understood in the averaged sense as described in section 3.5.1. In that case they are independent of the details of the dissipation process. In addition we obtained expressions (52) applicable to the case of very weak dissipation where resonant responses may occur. We also gave corresponding expressions (54) that can be used to evaluate the energy and angular momentum exchanged as a result of a flyby in a parabolic orbit. These can be applied to either the tidal capture problem or the problem of tidal circularisation at large eccentricity (see Ivanov & Papaloizou 2011).

Our expressions may be applied to either Sun-like stars with radiative cores and convective envelopes or more massive stars with convective cores and radiative envelopes. However, in this paper we considered only Sun-like stars in detail. We developed expressions for the overlap integrals in the low frequency limit, under the traditional approximation. These were evaluated analytically following a WKBJ approach that yielded eigenfrequencies and eigenfunctions for the rotationally modified gravity modes. Good agreement was found between analytic estimates of the overlap integrals, obtained from equation (119) making use of equations (113) and (117), and values obtained through direct numerical integration. These showed that for prograde forcing the overlap integrals increased as the rotation rate of the star increased and accordingly tidal interaction strengthened. The opposite, but more pronounced trend, was found for retrograde forcing (Figs 10-13). Knowledge of the overlap integrals and mode spectrum enables calculation of tidal energy and angular momentum exchange rates with the orbit as a function of tidal forcing frequency in the moderately viscous regime.

For Sun-like stars with inner radiative regions, our formalism allows one to find the tidal response of low frequency gravity modes without numerical evaluation of eigenfrequencies and overlap integrals, and thus find the orbital evolution of tidally interacting stars of this type. In order to implement it, it suffices to know the behaviour of the Brunt - Väisälä frequency close to the base of convection zone, the radius and the value of the density at that loca-

tion, as well as the integrals $I = \int_0^{r_c} \frac{dr}{r} N$, and, for rotating stars, the quantities determining the angular properties of the overlap integrals and given in Fig 1. In general, it is also necessary to know the decay rate of the WKBJ due to either dissipation or non-linear processes. Provided this information is given as a function of stellar age, one can construct more realistic models of tidal interaction, that allow for finite values of the eccentricity, a non zero inclination of the stellar rotation axis to the orbital plane, and stellar evolution by semi-analytic methods. It is important to generalise the analytic evaluation of the overlap integrals to apply to stars with different structure, eg. more massive stars with convective cores. This will be undertaken in future work.

It is interesting to note that in our formalism, the dependence of energy and angular momentum transfer rates on semi-major axis is determined by the dependence of the normalised overlap integrals \hat{Q} on the forcing frequency ω . The standard dependence (Zahn 1977) is recovered only when $\hat{Q} \propto \omega^{17/6}$, which may, in general, not be satisfied. Although in the models we consider in this paper, the dependence of \hat{Q} on ω is very close to a power law with index 17/6 in the limit $\omega \rightarrow 0$, it differs from this for short period orbits. It may also differ for stellar models with a different radial structure. This emphasises the necessity of analytical and numerical studies of the behaviour of the overlap integrals for stellar models with different masses and ages.

We used our formalism to determine the tidal evolution timescales for a binary consisting of a primary Sun-like star in which the tidal response was excited in the moderately large dissipation regime. Assuming synchronous rotation and that inwardly propagating rotationally modified gravity waves are dissipated close to the centre, the quantity, $T_e^{(1)}((1+\mu)/2)^{5/3}/\mu$, where $T_e^{(1)}$ is the circularisation time, was found to vary between $10^5 - 10^{10}$ yr. as the orbital period varied through the range 1 – 5d. The corresponding value of Q' defined by Goldreich & Soter (1966) varied through the range $3 \times 10^5 - 10^8$.

Similarly when the primary was not rotating, $T_e^{(2)}((1+\mu)/2)^{5/3}/\mu$, $T_e^{(2)}$ being the circularisation time in this case, was found to vary between $10^4 - 10^9$ yr. as the orbital period varied through the range 1 – 5d. The quantity Q' has a similar range of variation as is found in the synchronous case. In this non synchronous case, the quantity $T_a^{(2)}((1+\mu)/2)^{5/3}/\mu$ where the time scale for inspiral is $T_a^{(2)}$ varied between $10^6 - 10^{10}$ yr as the orbital period varied through the range 1 – 5d. Thus the latter process is unlikely to have the possibility of being effective for masses in the planetary regime ($\mu < 10^{-2}$) and orbital periods exceeding 2.5d.

In this paper when tackling the problem with periodic forcing potential we assume that the mode energy doesn't grow with time and the amount of energy supplied by tidal interaction is equal to the amount of dissipated energy. This condition may however be broken either very close to the resonances or in the case when the distance between two eigen frequencies is smaller than the inverse of a characteristic orbital evolution time. In the latter case the approach based on the Fourier transform may be more adequate as discussed in eg. Ivanov & Papaloizou (2004a).

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APPENDIX A: THE COEFFICIENTS $A_{m,k}$ FOR A COPLANAR SLIGHTLY ECCENTRIC ORBIT

Equation (6) gives the perturbing tidal potential acting on the primary component in the form

$$\Psi = r^2 \sum_{m,k} (A_{m,k} e^{-i\omega_{m,k} t} Y_2^m + \text{cc}) . \quad (\text{A1})$$

An expansion of the gravitational potential due to the perturbing secondary in spherical harmonics yields the coefficients $A_{m,k}$. We consider the case when the eccentricity of the binary orbit is small, such that powers of e higher than the second may be neglected, and when the rotational axis is perpendicular to the orbital plane. A convenient expression for the tidal potential that can be used to determine the $A_{m,k}$ under these conditions can for example be found in Zahn (1977). The non-zero coefficients are found to be

$$\begin{aligned} A_{0,0} &= \frac{GM_2}{2a^3} \sqrt{\frac{\pi}{5}} \left(1 + \frac{3}{2}e^2\right), & A_{0,1} &= \frac{3GM_2}{2a^3} \sqrt{\frac{\pi}{5}} e, \\ A_{0,2} &= \frac{9GM_2}{4a^3} \sqrt{\frac{\pi}{5}} e^2, & A_{2,1} &= \frac{GM_2}{4a^3} \sqrt{\frac{6\pi}{5}} e, \\ A_{2,2} &= -\frac{GM_2}{2a^3} \sqrt{\frac{6\pi}{5}} \left(1 - \frac{5}{2}e^2\right), & A_{2,3} &= -\frac{7GM_2}{4a^3} \sqrt{\frac{6\pi}{5}} e, \\ A_{2,4} &= -\frac{17GM_2}{4a^3} \sqrt{\frac{6\pi}{5}} e^2. \end{aligned} \quad (\text{A2})$$

Here M_2 is the mass of the secondary and a is the orbital semi-major axis. Note that for the purpose of evaluating the complex conjugate in (6) we have $A_{m,k}^* = A_{m,k} = A_{-m,-k}$. We also remark that in order to determine the rate of change of the semi-major axis correct to zero order in eccentricity and the rate of change of eccentricity correct to first order in eccentricity, terms $\propto e^2$ may be dropped when working with these coefficients.

APPENDIX B: COMPARISON OF ASYMPTOTIC EXPRESSIONS FOR TORQUES AND ENERGY DISSIPATION RATES GIVEN BY NORMAL MODE SUMS WITH THOSE OBTAINED USING RADIATION BOUNDARY CONDITIONS

Here we consider the response of a spherically symmetric stellar model with rotation under the traditional approximation. The governing equation for the linear response to a forcing potential $\Psi_{m,k} \exp(-i\omega_{m,k} t + im\phi)$ is

$$-\omega_{m,k}^2 \xi_{m,k} - 2i\omega_{m,k} (\Omega \cdot \hat{r}) \hat{r} \times \xi_{m,k} + \mathcal{C} \xi_{m,k} = -\nabla \Psi_{m,k}, \quad (\text{B1})$$

This follows from equation (5) provided Ω is replaced by its radial component $\Omega \cdot \hat{r}$. Making this replacement amounts to making the traditional approximation described in section 4. Of course when Ω is set to zero, the problem reduces to the problem of finding the response of a spherically symmetric non rotating star. Thus considering equation (B1) enables us to consider both rotating stars with the adoption of the traditional approximation and non rotating spherical stars.

The action of the operators \mathcal{C} and ∇ respectively on $\xi_{m,k}$ and $\Psi_{m,k}$ is as indicated by equation (15). It is taken as read that in addition to time dependence through a factor $\exp(-i\omega_{m,k} t)$, each of $\Psi_{m,k} \exp(im\phi)$ and $\xi_{m,k} \exp(im\phi)$ have separable angular dependencies related to the Hough functions for the specified m and with an appropriate value of Λ (see section 4). These dependencies will not be indicated below. We go on to discuss the response due to a single forcing term.

When the response has been obtained for a range of assumed angular dependencies corresponding to related values of Λ , a total response can be found using the orthogonality of the Hough functions (see section 4.4). Total torques or energy dissipation rates are then obtained by summing contributions associated with each value of Λ involved in the representation of the perturbing potential in terms of Hough functions. Note also that the values of Λ that are expected to dominate are associated with the potentially resonant mode spectrum, namely that associated with rotationally modified gravity modes, or simply g modes in the case of a non rotating star.

For simplicity of notation, the subscripts m, k on all quantities will be suppressed from now on. In addition we adopt spherical polar coordinates (r, θ, ϕ) for the purpose of discussion in this appendix.

B1 Torque evaluated with radiation boundary conditions

The equation to be solved is equation (B1). Noting the angular dependence indicated above, A solution can be found by the method of variation of parameters where we write

$$\xi = A(r) \xi_1 + B(r) \xi_2. \quad (\text{B2})$$

In the above ξ_1 is a solution of the homogeneous form of equation (B1) (Ψ set to be zero) that satisfies the correct boundary conditions enforcing regularity at $r = 0$ and ξ_2 is a solution of the homogeneous form of equation (B1) that satisfies the appropriate boundary conditions at the surface $r \rightarrow R_s$. Here we take the latter solution to have to correspond to an outgoing wave. In this case physical dissipation is supposed to take place for $r \sim R_s$. But note that we expect a corresponding parallel analysis to hold when ξ_1 is taken to correspond to an ingoing wave while regularity conditions are applied to ξ_2 for $r \rightarrow R_s$. In that case physical dissipation is supposed to take place for $r \sim 0$. In any case the function $A(r)$ is chosen to vanish for $r \rightarrow R_s$ and $B(r)$ is chosen to vanish for $r \rightarrow 0$. In addition the functions $A(r)$ and $B(r)$ satisfy the constraint

$$\xi_1 \cdot \nabla A(r) + \xi_2 \cdot \nabla B(r) = 0. \quad (\text{B3})$$

This ensures that derivatives of A and B do not appear when the pressure perturbation P' is evaluated (see below).

We remark that for the case when dissipation occurs near the surface considered here, when ω corresponds to a free oscillation eigenvalue, ξ_1 will correspond to the free normal mode. As we consider the asymptotic low frequency limit, where the spectrum is dense, and we assume a degree smoothness in the quantities of interest, ξ_1 may be assumed to correspond to the normal mode

with eigenfrequency closest to the forcing frequency. Thus for comparison with the modal summation approach described above, we identify ξ_1 with ξ_{j_0} . We further comment that the equilibrium tide is implicitly included in the decomposition (B2). But the important contribution for computing the net torque is the outward going wave component and the equilibrium tide does not affect this.

We make use of the expressions for $\mathcal{C}\xi$, $P'(\xi)$ and N^2 given by equations (55), (56) and (57) and we take the adiabatic exponent Γ_1 to be constant, for convenience, as in the main text. It follows from these that for any pair of solutions of the homogeneous form of equation (B1) such as ξ_1 and ξ_2 the latter equation implies that

$$W_{12} = \int_{S_r} (P'(\xi_2)\xi_1^* - P'(\xi_1^*)\xi_2) \cdot d\mathbf{S}, \quad (\text{B4})$$

where S_r is an interior spherical surface of radius r , is constant. In addition for the solution of the inhomogeneous equation of interest, we have

$$\int_{S \rightarrow S_{R_s}} (P'(\xi)\xi_1^* - P'(\xi_1^*)\xi) \cdot d\mathbf{S} - \int_V \rho \xi_1^* \cdot \nabla \Psi d\tau = 0. \quad (\text{B5})$$

The expression (B4) is associated with the conservation of wave action for freely propagating disturbances.

From equation (B1) we find that the total torque communicated to the forced body is given by

$$T_0(\xi) = -m\mathcal{I}m \left[\int_{S \rightarrow S_{R_s}} P'(\xi)\xi^* \cdot d\mathbf{S} \right]. \quad (\text{B6})$$

Using equation (B6) with the fact that the forced solution (B2) becomes $B\xi_2$ as the surface of the star is approached, the angular momentum transfer rate to oscillatory disturbances as a result of the tidal interaction may be evaluated as

$$T_0(\xi) = |B(R_s)|^2 T_0(\xi_2) \quad (\text{B7})$$

Furthermore, using (B5) we can express $B(R_s)$ in the form

$$B(R_s) = \frac{\int_V \rho \xi_1^* \cdot \nabla \Psi d\tau}{W_{12}} \quad (\text{B8})$$

Thus the tidally induced torque may be written as

$$T_0(\xi) = \frac{|\int_V \rho \xi_1^* \cdot \nabla \Psi d\tau|^2 T_0(\xi_2)}{|W_{12}|^2} \quad (\text{B9})$$

To relate this to expressions given in the main text, we now use the WKBJ approximation to find alternative expressions for some of the quantities in (B9).

WKBJ solutions

As follows from equation (87) in the main text after neglecting the 1 as compared to N^2/ω^2 in the last bracket on the left hand side, in the small ω^2 limit, free oscillations are governed by the following equation for $v = r^2 P^{1/\Gamma_1} \xi_r$, ξ_r being the radial component of the displacement,

$$\frac{\rho}{P^{2/\Gamma_1}} \frac{d}{dr} \left(\frac{\rho}{P^{2/\Gamma_1}} \frac{dv}{dr} \right) = -\chi^2 v \quad (\text{B10})$$

where

$$\chi^2 = \frac{N^2 \Lambda \rho^2}{\omega^2 r^2 P^{4/\Gamma_1}}. \quad (\text{B11})$$

We remark that the pressure disturbance is given by

$$P' = \frac{\omega^2 \rho}{\Lambda P^{1/\Gamma_1}} \frac{dv}{dr} \quad (\text{B12})$$

As follows from equation (88), away from turning points, the WKBJ solution for $v_1 = r^2 P^{1/\Gamma_1} \xi_{r,1}$, where $\xi_{r,1}$ is the radial component of ξ_1 , can be written in the form

$$v_1 = \frac{\cos \left(\int_{r_i}^r \Phi dr + \eta_1 \right)}{\chi^{1/2}}, \quad (\text{B13})$$

where $\Phi = N\sqrt{\Lambda}/(r\omega)$, r_i is the inner boundary radius of the wave propagation zone and η_1 is a constant phase determined by the boundary condition at $r = 0$.

The WKBJ solution for $v_2 = r^2 P^{1/\Gamma_1} \xi_{r,2}$, where $\xi_{r,2}$ is the radial component of the outgoing wave solution ξ_2 , takes the form

$$v_2 = \frac{\exp i \left(- \int_{r_i}^r \Phi dr + \eta_2 \right)}{\chi^{1/2}}, \quad (\text{B14})$$

where η_2 is another constant phase factor. We may use the above WKBJ solutions to evaluate the constant quantities $T_0(\xi_2)$ and W_{12} in the WKBJ approximation in the form

$$T_0(\xi_2) = \frac{m\omega|\omega|}{\Lambda}, \quad (\text{B15})$$

and

$$W_{12} = \frac{\omega|\omega|}{\Lambda} \exp i(\eta_2 - \eta_1 - \pi/2). \quad (\text{B16})$$

We remark that here we have assumed without loss of generality that the integration of the square of the angular factors over the spherical polar angles is normalised to unity. Using the above expressions in (B9) we obtain

$$T_0(\xi) = \frac{m\Lambda |\int_V \rho \xi_1^* \cdot \nabla \Psi d\tau|^2}{\omega|\omega|}. \quad (\text{B17})$$

The WKBJ spectrum

To proceed further we recall that from the free and undamped form of (B1)

$$\omega_j^2 \xi_j - \mathcal{C}\xi_j - \omega_j \mathcal{B}\xi_j = 0 \quad (\text{B18})$$

We use the fact that in the WKBJ limit the spectrum is given in the large j limit by the phase integral relation

$$\omega_j = \pm \frac{\sqrt{\Lambda} \int_{r_i}^{R_s} N r^{-1} dr}{j\pi} \equiv \omega_0/j, \quad (\text{B19})$$

where the \pm alternative corresponds to the positive and negative frequency modes respectively, and see equation (103) of the main text with $n \equiv j > 0$. Thus, noting that Λ is frequency dependent, we have

$$d\omega_j/dj \left(1 - \frac{\omega_j}{2\Lambda} \frac{\partial \Lambda}{\partial \omega_j} \right) = -\omega_j^2/\omega_0. \quad (\text{B20})$$

In order to re-express the above expression in terms of the mode norm, we begin by noting that if we replace N by $N(1 + \epsilon)$ in (B19) and regard ω_j to be a function of ϵ , we have

$$d\omega_j/d\epsilon|_{\epsilon=0} \left(1 - \frac{\omega_j}{2\Lambda} \frac{\partial \Lambda}{\partial \omega_j} \right) = \omega_j|_{\epsilon=0}. \quad (\text{B21})$$

However, the same result as (B21) can be obtained from (B18) by multiplying by $\rho \xi_j^*$ and integrating over the fluid volume and then treating the introduction of ϵ from the point of view of perturbation

theory. Noting that the operators are self adjoint, we find from this that

$$\begin{aligned} d\omega_j/d\epsilon|_{\epsilon=0} \int_V \rho (\omega_j^2 |\boldsymbol{\xi}_j|^2 + \boldsymbol{\xi}_j^* \mathcal{C} \boldsymbol{\xi}_j) d\tau \\ = 2\omega_j \int_V \rho N^2 |\boldsymbol{\xi}_j \cdot \hat{\mathbf{r}}|^2 d\tau \end{aligned} \quad (\text{B22})$$

Using the WKBJ solution (B13) to evaluate the right hand side of (B22), identifying ω and ω_j with ω_{j0} and $\boldsymbol{\xi}_1$ with $\boldsymbol{\xi}_{j0}$ and making use of (B20), (B21) and (B22) we obtain

$$d\omega_j/dj|_{j0} = -\frac{\omega_{j0}^3}{2N_{j0}\Lambda} \quad (\text{B23})$$

where we recall the definitions of the inner product (4) and norm (26).

Using (B23) together with (B17) enables the latter to be reduced to the form

$$T_0(\boldsymbol{\xi}) = m\omega_{j0} \frac{|\int_V \rho \boldsymbol{\xi}_{j0}^* \cdot \nabla \Psi d\tau|^2}{N_{j0} |d\omega_j(j)/dj|_{j=j0}}, \quad (\text{B24})$$

Recalling again the definition of the inner product this can be written in the alternative form

$$T_0(\boldsymbol{\xi}) = \frac{2\pi^2 m \omega_{j0} (\boldsymbol{\xi}_{j0}^* |\nabla \Psi|^2)}{N_{j0} |d\omega_j(j)/dj|_{j=j0}}, \quad (\text{B25})$$

The rate of energy dissipation is then

$$\dot{E}_c = -\omega_{j0} T_0(\boldsymbol{\xi})/m = -\frac{2\pi^2 \omega_{j0}^2 (\boldsymbol{\xi}_{j0}^* |\nabla \Psi|^2)}{N_{j0} |d\omega_j(j)/dj|_{j=j0}}, \quad (\text{B26})$$

This should be multiplied by a factor of two to account for the response to the complex conjugate of the applied forcing term as specified by for example by equation (6). This is seen from the form of (B6) to give an identical contribution. The result then becomes identical to a term corresponding to single values of m and k in the summation given by (45).